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NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the Secretary-Treasurer, W. D. Cairns, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-second Summer Meeting. The Association joins the American Mathematical Society in its Semi-centennial at Columbia University, New York, N.Y., September 6–9, 1938.

Twenty-third Annual Meeting, Richmond and Williamsburg, Va., December 27–31, 1938.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1938 and reported to the Secretary.

ALLEGHENY MOUNTAIN.

ILLINOIS, Carbondale, May 13–14.

INDIANA, Terre Haute, May 6–7.

IOWA, Sioux City, April 15–16.

KANSAS, Pittsburg, April 2.

KENTUCKY, Morehead, May 14.

LOUISIANA-MISSISSIPPI, Starkville, Miss., March 11–12.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 7; College Park, Md., December 10.

MICHIGAN, Ann Arbor, March 19.

MINNESOTA, Collegeville, May 14; Minneapolis, October 28

MISSOURI, Rolla, April 23.

NEBRASKA, Hastings, May 6.

OHIO, Columbus, March 31.

OKLAHOMA, Oklahoma City, February 11.

PHILADELPHIA, Collegeville, Pa., Nov. 26.

ROCKY MOUNTAIN, Boulder, Colo., April 15–16.

SOUTHEASTERN, Atlanta, Ga., April 1–2.

SOUTHERN CALIFORNIA, Claremont, March 26.

SOUTHWESTERN, Albuquerque, N.M., April 25–26.

TEXAS, Fort Worth, April 22.

WISCONSIN, West De Pere, May 14.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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THE MATHEMATICAL ASSOCIATION OF AMERICA

The following forty-six persons have been elected to membership in the Association on applications duly certified:

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SAM ZASLAVSKY, E.E.(C.C.N.Y.) Engineering draftsman, U. S. Navy Yard, Brooklyn, N. Y.

W. D. CAIRNS, *Secretary-Treasurer*

THE APRIL MEETING OF THE SOUTHEASTERN SECTINO

The sixteenth annual meeting of the Southeastern Section of the Mathematical Association of America was held at the Georgia School of Technology, Atlanta, Georgia, on Friday and Saturday, April 1–2, 1938. There were in attendance more than two hundred persons from fifty-five institutions, including the following seventy-three members of the Association: H. C. Ayres, D. H. Ballou, J. G. Barnes, D. F. Barrow, Helen Barton, W. S. Beckwith, C. W. Bruce, Iris Callaway, T. C. Carson, Edna J. Cofield, W. B. Coleman, H. M. Cox, Forrest Cumming, U. P. Davis, B. F. Dostal, F. G. Dressel, L. A. Dye, E. D. Eaves, Floyd Field, R. L. Fritz, H. K. Fulmer, Leslie J. Gaylord, A. M. Gignilliat, M. E. Gillis, J. T. Hains, C. L. Hair, W. T. Hanson, D. C. Harkin, R. A. Hefner, G. W. Hess, Ruby U. Hightower, P. R. Hill, H. K. Holt, J. A. Hyden, J. B. Jackson, Rosa L. Jackson, H. T. Karnes, F. W. Kokomoor, G. B. Lang, Curtis Ledford, F. A. Lewis, S. W. McInnis, A. N. McPherson, J. F. Messick, Verdie F. Miller, W. L. Miser, J. S. Morrel, F. D. Murnaghan, W. P. Ott, D. D. Peele, Z. M. Pirenian, R. A. Purviance, W. W. Rankin, B. P. Reinsch, G. E. Reves, H. A. Robinson, Douglas Rumble, J. A. L. Saunders, W. H. Sears, Jr., W. E. Sewell, T. M. Simpson, D. M. Smith, H. E. Spencer, A. L. Starrett, F. H. Steen, R. P. Stephens, Ruth W. Stokes, S. Helen Taylor, Louise Thompson, M. J. Turner, D. L. Webb, M. C. Wicht, F. L. Wren.

Sessions were held Friday afternoon and evening and Saturday morning. Professor J. B. Jackson, Chairman of the Section, presided, except Friday evening and part of Saturday morning when the Section was divided into subgroups according to the nature of the papers presented. Subgroups were presided over by Vice-Chairman J. A. Hyden, Iris Callway, T. M. Simpson and J. F. Messick. On Friday evening a joint dinner with the Georgia Academy of Science was held in honor of the visiting speaker, Professor F. D. Murnaghan of Johns Hopkins University. At this time Dean Floyd Field presided.

At the business session on Saturday the following officers were chosen for 1938–39: Chairman, J. A. Hyden, Vanderbilt University; Vice-Chairman, D. M. Smith, Georgia School of Technology; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The next meeting was scheduled for March, 1939, at The

Citadel in Charleston, S. C. A resolution was passed relative to the loss sustained by the Section in the passing of Professor J. D. Bond.

The following twenty-nine papers were presented:

1. "Some aspects of the teaching of mathematics in Japan and Korea" by Dr. W. P. Parker, Pyengyang, Korea, introduced by the Secretary.

2. "On the factorization of matric polynomials" by Professor H. S. Thurston, University of Alabama, introduced by the Secretary.

3. "Some inequalities connected with exponential functions" by Professor W. E. Sewell, Georgia School of Technology.

4. "Eighteenth century mathematics" by Professor W. W. Rankin, Jr., Duke University.

5. "The new Georgia high school requirement in mathematics" by Dean R. P. Stephens, University of Georgia.

6. "A class of real functions" by Professor J. M. Thomas, Duke University, by title.

7. "Moments of the point binomial" by Professor F. L. Wren, Peabody College.

8. "The basic ideas of arithmetic and algebra" by Professor F. D. Murnaghan, Johns Hopkins University.

9. "Certain formulas in compound interest and annuities" by Professor Z. M. Pirenian, University of Florida.

10. "A Cremona transformation of order 13" by Professor L. A. Dye, The Citadel.

11. "Some properties of the solution of linear difference equations" by Dr. H. C. Ayres, University of Alabama.

12. "Finite deformations of an elastic solid" by Professor F. D. Murnaghan, Johns Hopkins University.

13. "Conics in complex coördinates" by Professor Verdie F. Miller, Young Harris College.

14. "A study of the triangle by means of trilinear coördinates" by Professor G. W. Hess, Howard College.

15. "On volumes bounded by cylindrical surfaces" by Professor J. D. Mancill, University of Alabama, by title.

16. "Epitome equation of geometry" by Professor D. C. Harkin, Alabama Polytechnic Institute.

17. "Non-vanishing Laplace integrals" by Professor D. H. Ballou, Georgia School of Technology.

18. "Stieltjes integral equations" by Dr. F. G. Dressel, Duke University.

19. "The solution of a certain linear partial differential equation of first order" by Dr. L. D. Rodabaugh, University of Alabama, introduced by the Secretary.

20. "First lessons in definite integrals" by Professor W. L. Miser, Vanderbilt University.

21. "Functions of a generalized angle" by Dr. I. M. Hostetter, Howard College, introduced by the Secretary.

22. "Reduction of polynomial fractions to sums of partial fractions" by Professor F. H. Steen, Georgia School of Technology.

23. "A set of defining relations for the simple group of order 3420" by Professor F. A. Lewis, University of Alabama.

24. "A theorem on resultants of polynomials" by Professor J. S. Morrel, Vanderbilt University.

25. "Rings in many-valued logics" by Dr. D. L. Webb, Georgia School of Technology.

26. "Mathematical accuracy in a physical experiment" by Professor P. R. Hill, University of Georgia.

27. "Pretended accuracies in mathematical problems and computations" by Professor B. P. Reinsch, Florida Southern College.

28. "Measurement of student achievement in mathematics" by H. M. Cox, secretary to the Examiners, University System of Georgia.

29. "Discussion: high school requirements in mathematics" led by Professor D. M. Smith, Georgia School of Technology.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Dr. Parker showed how the Japanese and Koreans are overcoming certain difficulties in learning mathematics due to their past training such as would result from rote learning, a clumsy number system, and an over emphasis on classics.

2. Professor Thurston determined the number of ways in which a polynomial $F(X)$ may be factored as a product of linear factors, $X-X_i$, where the X_i are integral rational functions of a given matrix. The question of similar factorization was considered.

3. The exponential function may be defined as the limit of a binomial expansion, or as a Taylor expansion. The rates at which these expansions approach the function and each other were discussed by Professor Sewell.

4. Professor Rankin described the development of mathematics during the eighteenth century, showing that the builders of that period were mostly interested in the application of the new methods of analysis, analytical geometry, and calculus to the study of physical phenomena. The ideal rigor of the eighteenth century mathematicians was that of the ancient Greeks.

5. Dean Stephens discussed the new Georgia regulation for graduation from a senior high school which requires one unit in "general mathematics including the elements of algebra," but no geometry. From those preparing for college will come the future leaders. This special group should be given special consideration and should be advised to take more mathematics as a part of their preparation. Seemingly a great deal of experimentation is going on in many of the schools. This should be discouraged except in laboratory institutions. The required core program should consist of those subjects whose educational value has been thoroughly proved by past experience.

6. Let $f(x)$ be bounded and differentiable on the interval $a \leqq x \leqq b$. Let its

derivative satisfy $|f'(x)| \leqq M|f(x)|$, where M is a positive constant. Professor Thomas proved that if f has a zero on the interval, it is identically zero there. He applied the result to the uniqueness proof for the solution of systems of ordinary differential equations of the first order.

7. Professor Wren developed recurrence formulas which expressed for the point binomial, moments of any given order about q^n, p^n, and the true mean in terms of moments of the same type but lower order.

8. Professor Murnaghan discussed the properties of the positive integers which serve to define them and then passed to the fractions and negative numbers, pointing out the seeming paradoxes connected with them. The central position assumed by the commutative law of multiplication was emphasized and attention was called to the existence of non-commutative algebras and to their important applications in the domain of physics.

9. Professor Pirenian presented several formulas by which long divisions in the mathematics of finance may be replaced by multiplications, and long multiplications by additions or subtractions. These formulas were found useful in constructing certain tables and solving certain problems.

10. Professor Dye used two (1, 1) correspondences to generate this transformation. The first was between the quadrics of a pencil and the lines of a linear system of lines; the second related the conics cut from a quadric by planes through the associated line, to the points of a fixed line. The transformation was capable of generalization in two ways.

11. Dr. Ayres discussed the solution of a boundary value problem for an nth order nonhomogeneous linear difference equation. The boundary conditions were n linear relations involving differences of order $n-1$. A Green's function for the reduced equation was derived and led by means of a summation to a unique solution of the problem under consideration.

12. Professor Murnaghan gave an account of a recent theory of elasticity proposed by him in which the basic simplicity assumption of the classical theory of elasticity, namely, that the deformation is infinitesimal, is not made. The exact equations connecting the stress and strain tensors were given. The theory found confirmation by experiment when applied to the case of uniform hydrostatic pressure. Recent experiments by Bridgman on the compressibility of metals and of various rocks show that the theory predicts, to within the limits of experimental error, the experimental results over a range extending to a pressure of 45,000 atmospheres. A remarkable feature was that only one constant appeared in the formula.

13. Professor Miller gave the solution of problem 3789 of the MONTHLY (June–July, 1936) together with a generalization to all conics.

14. Professor Hess made a thorough study of the various properties of a triangle by means of trilinear coördinates, using the given triangle for reference.

15. The determination of certain volumes common to two given cylindrical surfaces was discussed by Professor Mancill. It was shown how a slight alteration of the usual procedure of the integral calculus greatly simplified the problem.

16. Professor Harkin gave various geometric and trigonometric interpretations to an identity attributed to Euler, but known in the tenth century by the Arab El-Sābī.

17. Professor Ballou gave a set of conditions on the determining function of a Laplace integral sufficient to ensure that the integral shall not vanish in the half-plane of convergence.

18. In this paper Dr. Dressel presented a short history of a Stieltjes integral equation, indicated when such an equation can be changed to an ordinary Fredholm equation, and finally pointed out some unsolved problems concerning the equation.

19. Dr. Rodabaugh proved an existence theorem for a solution of a linear partial differential equation in a simply connected region of the xy-plane.

20. Professor Miser exhibited several examples of definite integrals which can be evaluated as limits of sums. By such simple examples the students may learn to calculate areas, etc., as limits of sums and obtain a better comprehension of a definite integral.

21. The single parameter family of extremal arcs in space determined by a fixed vector at a point defines a Finsler space which may be considered as a generalization of a plane. It was the purpose of this note to find a measure of the generalized angle formed by two of the members of the family determining the plane, and to discuss functions of such an angle analogous to the trigonometric functions.

22. Professor Steen gave a simple method for obtaining the coefficients in any of the various types of partial fraction expansions.

23. Professor Lewis discussed the defining relations of the simple group of order 3420.

24. By means of linear dependence, necessary and sufficient conditions for the existence of common divisors of two polynomials were derived. Professor Morrel obtained formulas for such divisors in terms of certain minors of the conventional Sylvester determinant.

25. The rings which M. H. Stone has shown to exist in the case of Boolean algebra were extended by Dr. Webb to the algebra of many-valued logics.

26. Professor Hill discussed a physical experiment in which any desired degree of accuracy, however great, may be secured.

27. Professor Reinsch called attention to the frequent occurrence of errors in data and computation in science textbooks, and proposed remedies for the situation.

28. Mr. Cox presented a statistical study illustrating the achievement in mathematics of the freshmen and sophomore students of thirty Georgia colleges. The same battery of tests was used by a large number of Georgia high schools so that the sampling of students entering college can be analyzed.

29. Until the regulations for graduation from high school are changed, colleges will have to give special consideration to students entering with only one mathematics unit. It is imperative that those qualified to study mathematics,

not only be given an ample opportunity to do so, but be prevented from placing upon themselves a regrettable handicap by neglecting it.

H. A. ROBINSON, *Secretary*

THE MARCH MEETING OF THE MICHIGAN SECTION

The fifteenth annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, Michigan, on Saturday, March 19, 1938. The chairman of the Section, Professor V. G. Grove, presided. The Section was most fortunate in having among its attendants Professor E. J. Moulton, the invited speaker, and Professor W. D. Cairns, the national Secretary.

The attendance was about one hundred, including the following forty-seven members of the Association: N. H. Anning, W. L. Ayres, W. D. Baten, F. A. Beeler, W. M. Borgman, Jr., J. W. Bradshaw, J. B. Brandeberry, W. D. Cairns, R. V. Churchill, C. J. Coe, A. H. Copeland, Max Coral, S. E. Crowe, Wayne Dancer, P. S. Dwyer, C. H. Fischer, J. D. Fitzpatrick, W. B. Ford, R. E. Gaskell, T. N. E. Greville, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, E. E. Ingalls, L. S. Johnston, L. C. Karpinski, A. E. Lampen, Theodore Lindquist, C. E. Love, E. D. McCarthy, D. D. Miller, D. C. Morrow, E. J. Moulton, A. L. Nelson, L. F. Ollmann, L. C. Plant, J. E. Powell, G. Y. Rainich, E. D. Rainville, C. C. Richtmeyer, T. R. Running, M. E. Shanks, G. G. Speeker, T. O. Walton, Fern Welker, R. L. Wilder, Margarete C. Wolf.

Sixty-nine persons attended the noon luncheon at the Michigan Union. At the business meeting which followed, W. L. Ayres was elected chairman and P. S. Dwyer was elected secretary-treasurer. After some discussion it was decided to have but one meeting during the coming year.

The luncheon meeting was also addressed by Professor W. D. Cairns, who assisted the Section in its deliberations and indicated the rôle the Association should fill with reference to (a) a greater correlation with the interests of secondary school teachers, and (b) the preparation of good college teachers.

The following seven papers were presented at the morning and afternoon sessions:

1. "Angles between two flat subspaces in n-dimensions" by Professor G. Y. Rainich and R. S. Phillips, University of Michigan. Presented by Mr. Phillips.

2. "A certain class of variation problems" by Morris Friedman, Wayne University, introduced by Professor A. L. Nelson.

3. "The Fibonacci series and allied trigonometric identities" by Professor L. S. Johnston, University of Detroit.

4. "The solution of a problem in card matching" by Dr. T. N. E. Greville, University of Michigan.

5. "An algebra of Dedekindian numbers" by J. C. Rumball, Wayne University, introduced by Professor A. L. Nelson.

6. "Some remarks on maxima and minima" by Professor C. J. Coe, University of Michigan.

7. "Mathematical accuracy" by Professor E. J. Moulton, Northwestern University, by invitation.

Abstracts of these papers follow, the numbers corresponding to the numbers in the the list of titles:

1. Mr. Phillips gave a vectorial treatment of the problem of characterizing the relative position of two flat subspaces of a hyperspace. This had been treated analytically by Jordan and synthetically by Schoute. The method consists in finding stationary values of scalar products of pairs of unit vectors—one in each subspace. These quantities are obtained as the roots of a polynomial which appears as a determinant, some of whose elements are scalar products of unit vectors of the two subspaces.

2. In this paper Mr. Friedman discussed the calculus of variations problem involving the integral $I=\int_{x_0}^{x_1} y^{1/n}(1+y'^2)^m dx$ where m and n are real numbers such that $m>0$, $n<0$. He showed that there exist certain regions of the plane, depending on m and an arbitrarily chosen point P_0, the points of which can be joined to P_0 by one and only one extremal, and discussed properties of these extremals.

3. Professor Johnston exhibited several classical identities of trigonometry which have either exact or very closely related parallels in which the arguments are numbers of the Fibonacci sequence instead of trigonometric functions. He also exhibited several functional equations solvable either in terms of trigonometric functions of the arguments or in terms of the Fibonacci sequence which are identifiable by the same arguments. Also shown were some well known expressions for trigonometric functions of specific angles which are derivable from the same Fibonacci sequence.

4. Dr. Greville obtained the exact frequency distribution of matchings which occurs when two shuffled decks of cards, each consisting of five duplicates of each of five designs, are matched against each other, card by card. This paper has been published in the *Journal of Parapsychology*, Vol. II, No. 1, under the title, *Exact Probabilities under the Matching Hypothesis.*

5. Sections of an everywhere dense set of transfinite ordinal numbers were made defining a set of Dedekindian numbers. Mr. Rumball utilized the properties of the transfinite ordinals in establishing the corresponding properties of Dedekindian numbers. The method is analogous to the method used by Richard Dedekind in defining the real number system by cuts of the rational numbers.

6. Professor Coe discussed the problem of determining the extrema of a function $\phi(x, y)$ under the condition $f(x, y)=0$. The problem was reduced to finding those linear elements which are simultaneously tangent to some curve of the family $\phi(x, y)=$constant and to the curve $f(x, y)=0$. This reduction was shown to be a fertile source of problems on maxima and minima and to lead to an immediate solution of a large percentage of such problems ordinarily treated in the calculus.

7. To say that something is stated or proved with "mathematical accuracy" is usually considered the highest of praise. Professor Moulton undertook to illustrate how the efforts of mathematicians to attain precision have not been entirely successful. His examples were taken from elementary courses, referring in particular to definitions of a straight line and of a circle, to a method of finding points of intersections of curves, to the definition of sin x as used in trigonometry and in calculus, to a proof of a law of the mean, and to an application of Duhamel's Theorem. He advocated the use of more precise statements, but recognized the impossibility of attaining absolute precision, particularly in elementary courses.

P. S. DWYER, *Secretary*

THE FIFTH ANNUAL MEETING OF THE OKLAHOMA SECTION

The fifth regular meeting of the Oklahoma Section of the Mathematical Association of America was held at the Central High School in Oklahoma City, Oklahoma, on Friday morning, February 11, 1938, Professor H. L. Hall presiding.

The number in attendance was sixty-one, including the following twenty members of the Association: E. F. Allen, J. C. Brixey, J. H. Butchart, N. A. Court, R. C. Dragoo, W. V. N. Garretson, Esther Gassett, F. C. Gentry, Dot Jeannette Gifford, H. L. Hall, J. O. Hassler, E. E. Heimann, J. E. LaFon, Clarence McCormick, Dora McFarland, W. T. Short, C. E. Springer, E. B. Wedel, B. S. Whitney, J. H. Zant.

The following officers were elected for the next year: Chairman, W. T. Short, Oklahoma Baptist University; Vice-Chairman, J. H. Butchart, Phillips University; Secretary, C. E. Springer, University of Oklahoma. The next meeting will be held at Oklahoma City in February 1939 in connection with the Oklahoma Education Association.

The following papers were presented:

1. "Some recent advances in astrophysics" by Dr. H. S. Mendenhall, Oklahoma Agricultural and Mechanical College, introduced by Professor Allen.

2. "Quaternary Cremona groups of ternary type" by Professor F. C. Gentry, University of Oklahoma.

3. "M-points generated by similar triangles" by Professor J. H. Butchart, Phillips University.

4. "On numbers equal to the sum of their aliquot parts increased by the number of their aliquot parts" by Professor J. C. Brixey, University of Oklahoma.

5. "The right triangle as an aid in the teaching of integration" by Professor W. T. Short, Oklahoma Baptist University.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Dr. Mendenhall reviewed the developments in astrophysics of general

interest which have taken place during the last decade. He discussed quantitative, as well as qualitative, chemical analyses of the solar and stellar systems.

2. Professor Gentry considered the possibility of determining involutions in space by means of webs of quartic surfaces of degree two. Thirteen cases were found in which the base of the web contained simple base points and a double curve. In each case, by allowing the simple base points to vary while the remainder of the base was held fixed, these involutions were used to generate groups of Cremona transformations simply isomorphic to certain well known linear groups.

3. Professor Butchart proved for the sets of points (a_i), (b_i), (x_i), $(i=1, \cdots, n)$ that if the triangle $a_ib_ix_i$ is given in shape, then the centroids of the three sets determine a triangle of the same shape. Complex coördinates were used. Certain propositions from Johnson's *Modern Geometry* were proved as corollaries to the theorem.

4. Professor Brixey discussed the equation $S(n)=n$, where n is a positive integer and where $S(n)$ denotes the sum of the divisors of n, which are less than n, increased by the number of such divisors. Writing n in the form $n=2^\alpha p_1^{e_1} p_2^{e_2}, \cdots$, where $p_1, p_2, \cdots$ are distinct odd primes, he showed that α is odd and $e_1, e_2, \cdots$ are even. The only power of 2 which satisfies $S(n)=n$ is $n=2$, and n contains no $p_i \leqq 2^{\alpha+1}-1$. $S(n)\neq n$ if $n=2^\alpha p_1^{e_1}p_2^{e_2}$. The only solution less than 2,004,002 is $n=2$.

5. Professor Short exhibited several examples of integrals each containing one radical expression and showed the advantage of placing the radical expression on a suitably chosen side of a right triangle at the outset. The desired trigonometric substitution to effect the integration is then evident from the triangle.

C. E. SPRINGER, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The eighteenth regular meeting of the Southern California Section of the Mathematical Association of America was held at Pomona College, Claremont, California, Saturday, March 26, 1938. Professor A. D. Michal presided.

The attendance was about sixty, including the following twenty-six members of the Association: L. J. Adams, O. W. Albert, L. D. Ames, Harry Bateman, Clifford Bell, E. T. Bell, A. L. Buchman, Jessie R. Campbell, P. H. Daus, D. C. Duncan, Iva B. Ernsberger, Harriet E. Glazier, G. H. Hunt, C. G. Jaeger, Glenn James, G. R. Kaelin, I. Maizlish, A. D. Michal, W. B. Orange, Lena E. Reynolds, G. E. F. Sherwood, Marcus Skarstedt, C. W. Trigg, S. E. Urner, W. M. Whyburn, Euphemia R. Worthington.

The following officers were elected for the next year: Chairman, W. M. Whyburn, University of California at Los Angeles; Vice-Chairman, L. J. Adams, Santa Monica Junior College; Program Committee: Marcus Skarstedt, Whittier College, and Morgan Ward, California Institute of Technology. The meeting was tentatively scheduled for March 4, 1939, at Whittier College.

The following five papers were read:

1. "Mortality tables—their construction, graduation, and use" by A. L. Buckman, Occidental Life Insurance Co.

2. "Analysis of cosmic rays with the help of terrestrial and solar magnetic fields" by Professor P. S. Epstein, Professor of Theoretical Physics, California Institute of Technology, introduced by Professor Michal.

3. "Using the unusual—instructional values in the neglected exceptional cases, with examples" by Professor S. E. Urner, Los Angeles Junior College.

4. "Stability from the algebraic, analytical, and technical standpoint" by Professor Harry Bateman, California Institute of Technology.

5. "Some uses of matrix algebra in differential equations" by Professor W. M. Whyburn, University of California at Los Angeles.

Abstracts of the papers follow, the numbers corresponding to those in the list of titles.

1. Some applications of mathematics to the field of actuarial science were illustrated by Mr. Buckman, in his paper on the construction, graduation, and use of mortality tables. Formulas were given to obtain crude rates of mortality for population mortality tables and for mortality tables on insured lives. Comparison was made of various methods of graduation, including Orloff's weighting of crude data by probable errors, summation formulas, osculatory interpolation, Makeham's formulas and the Henderson-Whittaker Formula A. Some demographic interpretations of population mortality tables were given as well as the solution of a complicated tax problem requiring the determination of the value to each of five heirs of an estate left in trust.

2. Professor Epstein discussed the mathematical problems arising in connection with the motion of charged cosmic ray particles in the magnetic fields of the earth and sun. Particular attention was devoted to the question of the influence of the solar magnetic field upon the terrestrially observed phenomena, a subject recently advanced by Vallarta. The speaker pointed out that this cause might produce measurable diurnal variations of cosmic ray intensities in high magnetic latitudes and presented theoretically calculated curves for the magnitude of this effect. The comparison with the observations of Millikan and Neher (made in Saskatoon, Canada) was as yet inconclusive as the predicted differences lie on the verge of the experimental error.

3. Professor Urner believes: (1) The care used in setting up the conditions under which a theorem is true, is wasted unless the student sees clearly what could happen if the conditions were *not fulfilled*. (2) The proper frame of mind for apprehension of mathematical truth is one of *questioning*. (3) The exceptional cases constitute one of the most powerful tools for the creation of *interest*, in that they are likely to partake of the *spectacular*. Simple examples were listed, under such headings as equivalence of equations and systems of equations, linear dependence, intersections of curves whose equations are in polar coördinates, continuity, multiple-valued functions, existence of a derivative, implicit functions, end-points, maxima and minima, and graphical treatment of complex solutions of equations in two variables.

4. Professor Bateman first reviewed the theory of the stability of equilibrium largely from the historical standpoint, starting with the work of Huygens. Following Duhem the conditions for the stability of a floating body were obtained by considering all possible variations. Mention was made of recent results relating to the stability of ships. The algebraic problem connected with the stability of motion of a vehicle of transportation was discussed from a general standpoint and an account was given of some related developments. Some remarks were made also on the stability problems depending on the asymptotic forms of solutions of differential and integral equations. An attempt was made to indicate some of the branches of astronomy, biology, physics, and engineering in which problems of stability are of major importance.

5. The use of matrix algebra in studies of ordinary differential equations was discussed by Professor Whyburn. The major parts of the paper have appeared in an article by Birkhoff and Langer in the *Proceedings of American Academy of Arts and Science*, vol. 57, pp. 51–128, and in an article by the present author in the *American Journal of Mathematics*, vol. 56 (1934), pp. 587–592. Some applications of the notation in connection with Green's Function were discussed.

P. H. DAUS, *Secretary*

DETERMINATION OF A MAYAN UNIT OF LINEAR MEASUREMENT

G. F. CRAMER, Tulane University

The directors who were preparing for the Chicago Fair decided about 1930 to reproduce for the Fair the partly ruined buildings of the Nunnery Quadrangle at Uxmal in Northern Yucatan. The reproductions were to be constructed as nearly as possible exactly as the original buildings were when new. The careful measurements and scale drawings necessary for this work were made by members of an expedition working under the direction of the Department of Middle American Research of Tulane University.

This paper is concerned with the problem of using the measurements which appear on these plans to determine the size of the previously unknown unit of length which was used in the original structures. The measurements were arranged in order of size and all duplicates were discarded before the following work was started. The set of over a thousand dimensions appearing on the plans was thus cut down to 286.

The whole investigation was based on the assumption that any set of measurements of this sort would be almost certain to show a more frequent occurrence of multiples of the unit than of multiples of any other quantity not commensurable* with the unit. The set would, however, probably contain

* In this paper, two numbers are called "to all practical purposes not commensurable" if few or none of their common multiples lie within the range of the recorded measurements. The words commensurable and not commensurable were used in that sense throughout the paper.

many measurements which were not intended to be multiples of the unit and others which were inaccurately measured.

One plausible method of attacking the problem is that of dividing each measurement of the set by the integers 1, 2, 3, $\cdots$ and arranging the results in the form of a table with the original numbers in the first row, their halves in the second row, their thirds in the third row, and so on. Any entries appearing with unusual frequency would be suspected of being commensurable with the unit. However, this method is probably more suitable as an aid to guessing the unit than as a means of verifying the guess. In the present work, the above method was not used to justify any final conclusions, but the tabulation did show a remarkable number of entries which were almost exactly whole numbers of feet or whole numbers of inches, and this in spite of the fact that fractions of an inch were not dropped in making the measurements of the ruins.

The other method used requires a more extended discussion. Let us suppose that we are given a set of M numbers in which there is no particular predominance of multiples of any one quantity. If we plot these numbers on an X-axis, we shall find a more or less random distribution of points with no regular, periodic repetition of clusters of points. Let us now choose a number N of any convenient size and plot on the same axis the numbers kN for successive integral values of k. Next, we choose a number $2n$ which is less than N, and lay off a short interval of length n on each side of each of the division-points kN. Now, it seems reasonable that the number of points of the set M lying in one or another of these short intervals should be approximately $(2n/N)M$, since a point is, presumably, just as likely to fall in one position as in any other nearby position on the axis. We shall define the *predicted fraction* of the M numbers as $p=(2n/N)$.

If, on the contrary, our set of M numbers contains a great many which were intended to be multiples of some unit, we shall expect the plotted points to show a tendency to occur in regularly spaced clusters with centers at those points which represent the multiples of the unit which was used. In our practical application of this, we would probably find this clustering more noticeable if we did not discard all of the duplicate measurements, but it can be detected even in the smaller set of only 286 measurements. If N and n are chosen as before, we can determine how many of the numbers of the set M differ from multiples of N by an amount n or less, that is, how many of the points of the set lie in one or another of the short intervals previously referred to. Suppose an actual count shows m such numbers in the set M. We define the *actual fraction* of the M numbers as $P=m/M$. We also define the *ratio* as $R=P/p$, where p is the predicted fraction. It is clear that, if the chosen value of N is nearly the same as the unknown unit, then the corresponding value of R will be relatively high. If, on the contrary, N is not nearly commensurable with the unit, then the points which represent multiples of N will miss most of the clusters and the corresponding value of R will probably be less than one. In case the N is very nearly equal to the unit, we can show that the values of R which correspond to $N'=2N$ and

$N''=3N$ should be, roughly, the same as that of the R which was found for N. According to the statements above, we should expect a set of M random numbers to give values of R not greatly different from 1 for all N's of reasonable size, but should expect a set of M measured quantities to give unusually high values of R for some particular N_1 and its multiples. In the latter case, any N commensurable with N_1, but not a multiple of N_1 would be likely to give an R greater than 1, but not as great as the R's corresponding to N_1 and its multiples. Any N not commensurable with N_1 should, of course, give a value of R less than 1.

In the actual numerical work on the set of 286 numbers, the value $n=\frac{1}{4}$ inch was used because the men who made the measurements stated that their errors, particularly on the smaller measurements, were probably less than that. The value of R was calculated for each of a considerable number of different values of N chosen at rather short intervals. Since practical considerations seemed to indicate the necessity of some commonly used unit less than a foot, no values of N greater than 12 inches were tried. The results of this work as shown in Table I seem to show that the most likely unit is very close to 3 inches. We notice that the R's which correspond to the values 3, 6, 9, and 12 of N are considerably larger than any of the other R's. The values of R which correspond to values of N which are commensurable with 3 but not divisible by 3 are also somewhat high, but, as was mentioned above, this was to be expected. The 3 was chosen rather than the 6, 9, or 12 on account of some facts which are to be mentioned in the conclusion.

TABLE I

N	$\frac{3}{4}$	1	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$	$2\frac{3}{4}$	3	$3\frac{1}{4}$
R	0.97	1.29	1.14	0.85	1.39	0.94	0.93	1.00	1.48	0.97
N	$3\frac{1}{2}$	$3\frac{3}{4}$	4	$4\frac{1}{2}$	5	6	7	8	9	10	12
R	0.78	1.13	1.33	1.19	1.19	1.58	.98	1.28	1.57	1.26	1.68

The small number of measurements in the set M made it advisable to try the above process on a set of 286 telephone numbers. The last two digits of each number were pointed off for decimal places, and the other work was done in the same way as before. The results appear in Table II. We notice that the values of R do not differ from 1 by nearly so much as was the case in the previous work. The values of R are, in this case, much more nearly what we should expect from a set of random numbers.

TABLE II

N	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
R	0.95	0.89	1.06	0.92	0.98	1.07	1.28	0.82	0.80
N	6	7	8	9	10	12			
R	0.88	1.12	1.06	0.94	0.84	1.01			

As a check of the validity of the process used, the same method was used on a

set of 286 measurements of door and window openings, cut stones, steps, and many other objects about the campus at Tulane. All of these last measurements were made in centimeters without observing the equivalent measurement in feet and inches at all. The objects used were, it is true, objects which probably had to be measured in some way during their construction, but there was never any intentional selection of objects whose dimensions would be likely to give favorable results. Furthermore, the plans used in computing Table I contained many measurements of similar objects. In Table III, the value of n was $\frac{1}{4}$ cm. This table shows a very marked tendency to emphasize the multiples of 1 inch and of 2 inches in our modern construction. The values 2.54 and 5.08 of N gave values of R which are of the same order of size as those found in Table I for $N=3$ and its multiples. In Table III, however, the other values of N which were used were, to all practical purposes, "not commensurable" with 2.54 and hence there were no other very high values of R occurring in the table.

TABLE III

N	1	$1\frac{1}{2}$	2	2.54	3	$3\frac{1}{2}$	4	5	5.08
R	0.96	1.04	0.98	1.57	0.98	1.03	0.86	1.15	1.81
N	6	7	8	9	10				
R	0.75	1.03	0.54	0.88	1.19				

In conclusion, it should be mentioned that these Mayan buildings from which Table I was computed showed frequent repetition of 5′0″. Since this is 20 times 3″, and since the Mayan number system used the base 20, it seems quite likely that there was a large unit of 60″ as well as the smaller unit. If this is true, then it is clear that our 3″ estimate could hardly be in error by as much as 1/20 in. Our throwing away all duplicate measurements undoubtedly weakened the results, since many multiples of 3″ were repeated again and again in the plans. Moreover, we could have gotten larger values of R than we did by using the value 1/8 instead of the value $\frac{1}{4}$ for n, as there were relatively few of the measurements which involved eighths. The method worked fairly well in spite of our having chosen the least favorable conditions. The 3″ unit seems more likely than the 6″, 9″, or 12″ unit, first because of its more convenient size, and second because it would be rather hard to explain the very high value of R for $N=3$ on any other grounds. It is not likely that halves and thirds of a unit would be as popular among workmen as whole numbers would be. There was quite frequent repetition, however, of the measurement $\frac{3}{4}$″. This did not result in a correspondingly high value of R for that value of N because hardly any other odd multiples of $\frac{3}{4}$″ occurred at all and duplicates were not counted.

THE ESTIMATION OF THE TOTAL FISH POPULATION OF A LAKE

ZOE EMILY SCHNABEL, University of Wisconsin

The purpose of this note is to discuss and compare as to fundamental assumptions four different methods which have been developed for the estimation of the fish population of a given lake from a sample census. Although the solutions outlined are adaptations of well known statistical procedures, it has seemed worthwhile to point out their application to this particular problem of estimation in the hope that the genéral methods may prove useful in other types of population studies.

In order to obtain the sample fish census, which may or may not be confined to a particular species, seining stations are established at various points on the given lake, samples taken at periodic intervals (usually about 24 hours), the fish tagged and replaced. Records are kept for each sample of the total number, the number of new fish, and the number of recaptures. If it be assumed that each sample be drawn at random and that the population of the lake remain constant throughout the period under consideration, it is possible to obtain an estimate of the total number of fish in the lake from the data of the census.

If N denote the total population of the lake, M_i the number of tagged fish in the lake when the ith sample is drawn, t_i the total number, r_i the number of recaptures, and d_i the number of untagged fish in the ith sample, then M_i/N is the probability of drawing a marked fish in a single trial in the ith sample, and, by the binomial distribution,

$$p_i = C_{r_i}^{t_i}\left(\frac{M_i}{N}\right)^{r_i}\left(1 - \frac{M_i}{N}\right)^{d_i}$$

is the probability of drawing r_i recaptures and d_i new fish in the ith sample. It follows that $P=\prod_{i=1}^{n} p_i$ is the probability of obtaining the exact figures that were obtained in a census of n samples. The method of maximum likelihood,* which has been applied to this expression by Evelyn Hull and M. H. Ingraham, gives as an estimate of N the positive real root of the nth order equation

$$\sum_{i=1}^{n} \frac{d_i M_i}{N - M_i} = \sum_{i=1}^{n} r_i,$$

which can be expanded in the form

$$\sum_{i=1}^{n} \frac{d_i M_i}{N}\left(1 + \frac{M_i}{N} + \cdots\right) = \sum_{i=1}^{n} r_i. \tag{1}$$

Approximations to the root N may be obtained by taking a finite number of terms in (1). If N_k be the positive root of the equation using k terms of the left

* Fisher, R. A., On the mathematical foundations of theoretical statistics, Philosophical Transactions of the Royal Society of London, series A, vol. 222, pp. 309–368.

member, then the sequence $\{N_k\}$ is a monotonic increasing sequence bounded above by N. The accuracy with which N_k approximates the root N can be estimated by calculating an upper bound of N which is dependent upon N_k.

The remainder after k terms of the series $\sum_{r=0}^{\infty}(M_i^r/N^r)$ is

$$R_k = \frac{M_i^{k-1}}{N^{k-1}} \frac{M_i}{N - M_i}.$$

Since $N_k < N$ and $M_i \leqq M_n$,

$$R_k < \frac{M_i^{k-1}}{N^{k-1}} \frac{M_i}{N_k - M_i} \leqq \frac{M_i^{k-1}}{N^{k-1}} \frac{M_n}{N_k - M_n}$$

provided k be chosen large enough that $N_k > M_i$. Now (1) can be written in the form

$$(\sum r_i)N^k - (\sum d_iM_i)N^{k-1} - \cdots - \sum d_iM_i^k - N^{k-1}\sum d_iM_iR_k = 0,$$

and it follows that an upper bound for N is given by the positive root of

$$\text{(1a)} \quad (\sum r_i)N^k - (\sum d_iM_i)N^{k-1} - \cdots - \sum d_iM_i^k - \sum\left(\frac{d_iM_i^{k+1}}{N_k - M_i}\right) = 0,$$

and a greater upper bound by the positive root of

$$\text{(1b)} \quad (\sum r_i)N^k - (\sum d_iM_i)N^{k-1} - \cdots - \sum d_iM_i^k - \frac{M_n}{N_k - M_n}\sum d_iM_i^k = 0.$$

Since all terms except the last in each of these equations are involved in the equation from which N_k is obtained, very little additional computation is necessary in finding these bounds of N after N_k has been determined. The roots of (1a) and (1b) decrease as k increases so that the required root N of (1) can be fixed between satisfactorily narrow limits by taking k large enough. In actual practice, it is usually sufficient to find N_3 and the corresponding upper bound from (1b). The computational labor in obtaining approximations higher than N_3 is often prohibitive.

Certain special cases of (1) are of interest. If the equation be written in the form

$$\sum_{i=1}^{n} \frac{t_iM_i - r_iN}{N - M_i} = 0,$$

it is apparent that if M_i is negligible compared with N the root of (1) is approximately

$$\text{(2)} \qquad \frac{\sum_{i=1}^{n} t_iM_i}{R},$$

where $R=\sum_{i=1}^{n} r_i$. Furthermore, if $M_i = M$ for all i,

$$N = M\frac{\sum t_i}{\sum r_i}. \tag{3}$$

This formula is applicable to the data of experiments in which the number tagged is held constant after a certain point. The method has the disadvantage that the data taken before M becomes constant are not utilized.

A second method suggested to me by M. H. Ingraham can be applied when M_i/N is small and r_i may be assumed to follow the Poisson law of distribution. In such cases,

$$P = \prod_{i=1}^{n} \frac{\left(\frac{M_i t_i}{N}\right)^{r_i} e^{-M_i t_i/N}}{r_i!}$$

is the probability of obtaining r_i recaptures in a total of t_i for all i. We find that P is a maximum when

$$N = \frac{\sum_{i=1}^{n} t_i M_i}{R},$$

a result which agrees with (2). It is interesting that this formula can also be derived from a simple consideration of the expected number of recaptures in the ith sample,

$$\epsilon(r_i) = \frac{t_i M_i}{N}, \qquad (i = 1, 2, \cdots, n).$$

The expected number in n samples is

$$\sum_{i=1}^{n} \epsilon(r_i) = \sum_{i=1}^{n} \frac{t_i M_i}{N},$$

and, if $\epsilon(r_i)$ is replaced by r_i, formula (2) results.

A fourth estimate of N can be obtained by minimizing χ^2, the Pearson measure of goodness of fit.* By definition,

$$\chi^2 = \sum_{i=1}^{n} \frac{(m_i' - m_i)^2}{m_i}$$

where m_i' is the observed and m_i the expected frequency in the ith sample. It is evident that χ^2 is a measure of the agreement between observation and theory. The expected number of recaptures in the ith sample is $t_i M_i/N$ so that, in this case,

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* Pearson, Karl, Philosophical Magazine, 5th series, vol. 50, 1900, pp. 157–175.

$$\chi^2 = \sum_{i=1}^{n} \frac{\left(\frac{t_i M_i}{N} - r_i\right)^2}{\frac{M_i t_i}{N}}.$$

This is a minimum when

$$(4) \qquad N = \left[\frac{\sum_{i=1}^{n} M_i t_i}{\sum_{i=1}^{n} \left(\frac{r_i^2}{M_i t_i} \right)} \right]^{1/2}.$$

Since the χ^2 test is based on the assumption of fairly large cell frequencies, this method will not give reliable results if r_i is small.

The formulas developed here have been applied to various types of data with quite consistent results. We shall examine in particular three sets of data which exhibit markedly different characteristics. Census I concerns the number of black crappies in a large lake (5,600 acres) and is taken from unpublished data by D. H. Thompson of the Illinois State Natural History Survey. In Census II, which was made by C. Juday and C. L. Schloemer on a lake of about 95 acres, all species of fish were considered. Census III represents the results of a bean-drawing experiment conducted by the author. The main features of these three groups of data will be apparent from the following table:

Census Number	I	II	III
Number of Samples	79	39	14
Total in All Samples	15,284	10,033	2,055
Total Recaptures	81	2,176	767
Total Tagged	10,986	4,567	1,288

The data of Census III were such that the value of χ^2 based on the deviations of the observed from the expected number of recaptures would be exceeded 80% of the time in random drawings. Therefore, the data could reasonably be assumed to fulfill the hypothesis of random sampling.

When the methods of this paper were applied to these sets of data, the following results were obtained:

Formula	Census I	Census II	Census III*
1	$N_3 = 1,300,542$	$N_3 = 15,084$	$N_5 = 1862$
1a		$N < 15,345$	$N < 1915$
1b	$N < 1,300,543$	$N < 15,347$	$N < 1932$
2	1,300,276	15,307	1895
3	1,252,822	17,579	1963
4	734,755	14,093	1886

* The actual number of beans was 1872.

It is seen that N_3 is an excellent approximation to the root N of (1) for Census I and that the results of (1) and (2) agree closely in this case; this is due to the fact that M_i/N is small. N_3 and the corresponding bound of N computed from (1b) prove sufficient for the data of Census II. The more refined methods are necessary for Census III, since drawing was continued until M_i/N exceeded 0.6. Formula (3) can be applied to only a small part of the data in each set (19 samples in Census I and 3 samples in Census II) so that the results do not agree closely with those of other methods. The low results of formula (4) when applied to I and II are due to the fact that the hypothesis of a fairly large number of recaptures in comparison with the total is not satisfied. It is interesting that the deviations of the estimates of N from the true value in Census III, which most nearly satisfies the hypotheses under which the formulas were developed, range from 0.55% to 5% of the total.

Since the assumptions of random sampling and constant population are only rough approximations to the actual situation in taking a sample fish census, small differences between the results of the various methods are not important. The maximum likelihood solution has certain theoretical advantages and is applicable to a wide variety of data, but the other formulas give useful estimates when the data are such as to warrant their use. It should be emphasized, however, that none of the solutions can be expected to provide more than an estimate of the general order of magnitude of the total population.

CONTINUED FRACTIONS AND MODIFIED CONTINUED FRACTIONS FOR CERTAIN SERIES

J. W. BRADSHAW, University of Michigan

1. Introduction. Important in the study of series whose general terms are rational functions of the index is that particular series of this sort whose terms are the reciprocals of positive integral powers of the terms of an arithmetical progression, $a, a+d, a+2d, \cdots, a+nd, \cdots$, the series $\sum_{n=0}^{\infty}(a+nd)^{-t}$ and the alternating series $\sum_{n=0}^{\infty}(-1)^n(a+nd)^{-t}$, where t is a positive integer [1].* Convergent series of this type converge but slowly, and the problem of increasing their rate of convergence is one which has interested mathematicians for a century. The method for transforming series set forth by Kummer in 1837 [2], now commonly known by his name, has been the subject of numerous memoirs since that time [*e.g.*, **3, 4**].

In a number of articles written by Glaisher about 1903 [5], he was concerned with this same problem of improving the convergence of certain series, but the method that he used was quite different. By adding to a given series term by term another series whose sum was known he obtained an extensive collection of equivalent series. Even as recently as 1933 the problem has engaged the attention of Shohat [6].

* See references at the end of the paper.

In 1912 the present author, following the method suggested by Glaisher, showed that a sequence of more and more rapidly convergent series could be found for $\pi/4$, whose first terms approached the form of an infinite product [**7**]. The attempt to extend this result to more general series has led to certain continued fractions and to certain other forms which may be called modified continued fractions. The latter results appear as infinite continued fractions having an obvious and simple law of formation, but with this modification, that if it is desired to break off at any point, the last partial fraction is replaced by a different one.

2. The alternating series. We first consider the alternating series $\sum_{n=0}^{\infty}(-1)^n(a+nd)^{-t}$, a and d real, t a positive integer. To put this in simpler form we take out the factor $(2/d)^t$ and replace $2a/d$ by x. The series becomes $(2/d)^t\sum_{n=0}^{\infty}(-1)^n(2n+x)^{-t}$ and the problem is reduced to the consideration of the series $\sum_{n=0}^{\infty}(-1)^n(2n+x)^{-t}$, a function of the variable x, which we denote by $\phi_t(x)$. We shall assume for convenience that x lies in the interval $2 \leqq x \leqq 4$. This will not prove a limitation, since any series of this form, by the addition or subtraction of a finite number of terms, may be made to depend on the value of $\phi_t(x)$ in this interval. Thus the series $1-2^{-t}+3^{-t}- \cdots$ and the series $1-3^{-t}+5^{-t}- \cdots$ appear, respectively, as $2^t\phi_t(2)$ and $1-\phi_t(3)$.

From the series

$$\phi_t(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+x)^t} \tag{1}$$

we subtract term by term the series

$$\sum_{n=0}^{\infty} (-1)^n \left[\frac{g(2n+x-1)}{2f(2n+x-1)} + \frac{g(2n+x+1)}{2f(2n+x+1)}\right] = \frac{g(x-1)}{2f(x-1)}, \tag{2}$$

where $f(x)$ and $g(x)$ are polynomials of degree r and $r-t$, respectively, each containing either only odd powers or only even powers of x:

$$\begin{cases} f(x) = x^r + p_2x^{r-2} + p_4x^{r-4} + \cdots \begin{cases} + p_r, & \text{if } r \text{ is even,} \\ + p_{r-1}x, & \text{if } r \text{ is odd,} \end{cases} \\ g(x) = x^{r-t} + q_2x^{r-t-2} + q_4x^{r-t-4} + \cdots \begin{cases} + q_{r-t}, & \text{if } r-t \text{ is even,} \\ + q_{r-t-1}x, & \text{if } r-t \text{ is odd.} \end{cases} \end{cases} \tag{3}$$

The coefficients $p_2, p_4, \cdots$ and $q_2, q_4, \cdots$ are to be chosen so as to reduce as far as possible the degree in n of the resulting numerator,

$$\begin{aligned} F_t(2n+x) = {} & 2f(2n+x-1)f(2n+x+1) \\ & - (2n+x)^t[g(2n+x-1)f(2n+x+1) \\ & \qquad + g(2n+x+1)f(2n+x-1)]. \end{aligned} \tag{4}$$

Replacing $2n+x$ by s, it is readily seen that the coefficient of s^{2r} in

$$F_t(s) = 2f(s-1)f(s+1) - s^t[g(s-1)f(s+1) + g(s+1)f(s-1)] \tag{5}$$

vanishes identically, and that since $F_t(-s) \equiv F_t(s)$, $F_t(s)$ contains only even powers of s.

By equating to zero in succession the coefficients of $s^{2r-2}, s^{2r-4}, s^{2r-6}, \cdots$ in $F_t(s)$ the following equations are obtained, involving the so-called Eulerian numbers, which may be defined by the symbolic equation [**8**, pp. 25 and 458]

$$(E+1)^\nu + (E-1)^\nu = \begin{cases} 0, & \nu > 0, \\ 2, & \nu = 0, \end{cases} \qquad E_0 = 1.$$

The Eulerian numbers of odd index are zero, the first few of even index are $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, $E_8 = 1{,}385$, $E_{10} = -50{,}521$, $E_{12} = 2{,}702{,}765$, $E_{14} = -199{,}360{,}981$, $E_{16} = 19{,}391{,}512{,}145$. Early calculation of these numbers was extended by Glaisher [**9**] as far as E_{54} and by Joffe to E_{100} [**10**].

$$(6) \qquad \begin{cases} p_2 - q_2 = -\binom{t+1}{2} E_2, \\ p_4 - q_4 = -\binom{t+1}{2} E_2 p_2 - \binom{t+3}{4} E_4, \\ p_6 - q_6 = -\binom{t+1}{2} E_2 p_4 - \binom{t+3}{4} E_4 p_2 - \binom{t+5}{6} E_6, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

Since these equations are independent of r they may be considered as continuing indefinitely. From this point on we shall have to treat separately the different cases for t.

3. The series $\phi_1(x)$. This function is $\frac{1}{2}\beta(x/2)$, where $\beta(x)$ is the so-called "small Beta function" [**11**].* We shall treat the case for which r is even; that for which r is odd may be handled in a similar fashion and the results for both cases can be combined in a single formula.

The $r-1$ equations obtained by equating to zero the coefficients of $s^{2r-2}, \cdots, s^2$ in $F_t(s)$ now assume for $F_1(s)$ the form

$$(7) \qquad \begin{cases} p_2 - q_2 = -E_2, \\ p_4 - q_4 = -E_2 p_2 - E_4, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ p_{r-2} - q_{r-2} = -E_2 p_{r-4} - E_4 p_{r-6} - \cdots - E_{r-2}, \\ p_r = -E_2 p_{r-2} - E_4 p_{r-4} - \cdots - E_r, \\ 0 = -E_2 p_r - E_4 p_{r-2} - \cdots - E_{r+2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = -E_{r-2} p_r - E_r p_{r-2} - \cdots - E_{2r-2}. \end{cases}$$

* But practice is not uniform, Nörlund calls this function $\frac{1}{2}g(x)$ [**8**, p. 100].

If the coefficients $p_2, p_4, \cdots, p_r$ are chosen to satisfy the last $\frac{1}{2}r$ of these equations and the coefficients $q_2, q_4, \cdots, q_{r-2}$ to satisfy the remaining $\frac{1}{2}r-1$, $F_1(s)$ will be independent of s and hence equal to $2f(-1)f(1)$. This gives the result

$$(8)\quad \phi_1(x) - \frac{g(x-1)}{2f(x-1)} = \sum_{n=0}^{\infty}(-1)^n \frac{f(-1)f(1)}{(2n+x)f(2n+x-1)f(2n+x+1)}.$$

The following table gives the results of this calculation for $r \leqq 8$:

r	1	2	3	4	5	6	7	8
p_2		1	5	14	30	55	91	140
q_2			4	13	29	54	90	139
p_4				9	89	439	1,519	4,214
q_4					64	389	1,433	4,079
p_6						225	3,429	24,940
q_6							2,304	21,365
p_8								11,025

Thus, for example, using $r=4$ we have

$$\begin{cases} \phi_1(x) = \dfrac{(x-1)^3+13(x-1)}{2[(x-1)^4+14(x-1)^2+9]} \\ \qquad + \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{(4!)^2}{(2n+x)[(2n+x-1)^4+14(2n+x-1)+9] \cdot [(2n+x+1)^4+14(2n+x+1)+9]}, \end{cases}$$

so that our original series is replaced by one whose general term has a constant numerator, while the denominator is a polynomial of ninth degree in n.

4. Rational approximations. It may be observed* that the polynomials $f(x)$ take on remarkable values for the odd integers 1, 3, 5, $\cdots$. Thus $f(1)=r!$, $f(3)=(2r+1)r!$, $f(5)=(2r^2+2r+1)r!$, $f(7)=1/3(4r^3+6r^2+8r+3)r!$, and $f(9)=1/3(2r^4+4r^3+10r^2+8r+3)r!$

From this it appears that the external term $\frac{1}{2}g(x-1)/f(x-1)$, taken without the series, constitutes an approximation to the function $\phi_1(x)$ in the interval $2 \leqq x \leqq 4$. For the first term of the series has its largest value in this interval for $x=2$, and for this value it takes on the form $\frac{1}{2}f(-1)f(1)/[f(1)f(3)]=1/[2(2r+1)]$, which approaches zero as r increases indefinitely.

The sequence of these external terms for successive values of r are successive convergents of an infinite continued fraction, which in the notation adopted by Perron [**12,** p. 3], leads to the equation

$$(9)\quad 2\phi_1(x) = \frac{1|}{|x-1} + \frac{1|}{|x-1} + \frac{4|}{|x-1} + \frac{9|}{|x-1} + \frac{16|}{|x-1} + \frac{25|}{|x-1} + \cdots.$$

---

* These conclusions, like most of the results in this article, have been obtained by induction: I have not yet obtained a rigorous proof.

This yields, for $x=2$,

$$\log 2 = 2\phi_1(2) = \frac{1|}{|1} + \frac{1|}{|1} + \frac{4|}{|1} + \frac{9|}{|1} + \frac{16|}{|1} + \cdots,$$

a relation known to Euler [**12,** p. 208], and, for $x=3$,

$$\frac{\pi}{4} = 1 - \phi_1(3) = 1 - \frac{1}{2}\left\{\frac{1|}{|2} + \frac{1|}{|2} + \frac{4|}{|2} + \frac{9|}{|2} + \frac{16|}{|2} + \cdots\right\}.$$

Sequences of rational functions which are better approximations to the function $\phi_1(x)$ may be found by combining with the external term one or more terms of the series in the sequence of equations (8). Using the first term only we have the approximation

$$\frac{g(x-1)}{2f(x-1)} + \frac{f(-1)f(1)}{xf(x-1)f(x+1)}.$$

However, since the first term in the series (8) was obtained by subtracting the first term of (2) from the first term of (1), this can be put in the simpler form

$$\frac{1}{x} - \frac{g(x+1)}{2f(x+1)}.$$

Setting aside the factor $1/2x$, this formula gives the sequence of rational fractions

$$\frac{x+2}{x+1}, \quad \frac{x^2+3x+4}{x^2+2x+2}, \quad \frac{x^3+4x^2+11x+12}{x^3+3x^2+8x+6}, \cdots.$$

An obvious and very simple law of formation of these is revealed when they are turned into continued fractions; they are all represented by the form

$$\left[1, x, \frac{x}{2}, \frac{x}{3}, \cdots, \frac{x}{r-1}, \frac{x+r}{r}\right], \tag{10}$$

using again a notation of Perron [**12,** p. 27], in this case the symbol $[b_0, b_1, b_2, \cdots]$, for the continued fraction whose partial numerators are all 1, and whose partial denominators are $b_1, b_2 \cdots$. This suggests an infinite continued fraction, but with this significant modification, that if one wishes to break off at any particular point and obtain an approximation, the last partial denominator x/r is to be replaced by $(x+r)/r$. The conclusion is that $\phi_1(x)$ can be represented approximately by

$$\frac{1}{2x}\left[1, x, \frac{x}{2}, \frac{x}{3}, \cdots, \frac{x}{r-1}, \frac{x+r}{r}\right]. \tag{11}$$

Euler's continued fraction for $\pi/2$,

$$1+\frac{1|}{|1}+\frac{1\cdot 2|}{|1}+\frac{2\cdot 3|}{|1}+\frac{3\cdot 4|}{|1}+\cdots,$$

is equivalent to the suggested infinite continued fraction for $x=1$ [**12,** p. 213].

If in the same way two terms of the series are used, the formula to be employed is

$$\frac{1}{x}-\frac{1}{x+2}+\frac{g(x+3)}{2f(x+3)}.$$

After setting aside the factor $1/2x$ the resulting rational fractions can be comprehensively represented by the formula

$$\left[1,\ x,\ \frac{x}{2},\ \frac{x}{3},\ \cdots,\ \frac{x}{r-1},\ \frac{x-r}{r},\ \frac{x+4r+2}{8r+4}\right], \tag{12}$$

suggesting the same infinite continued fraction, but with a double modification.

The use of three terms of the series leads to the approximate representation of $\phi_1(x)$ by

$$\frac{1}{2x}\left[1,\ x,\ \frac{x}{2},\ \frac{x}{3},\ \cdots,\ \frac{x}{r-1},\ \frac{x+r}{r},\ -\frac{x+4r+6}{8r+4},\ -\frac{(2r+1)x}{(2r+1)^2+1}\right]. \tag{13}$$

5. Estimating the approximation. The process by which the continued fractions were arrived at suggests an immediate means of estimating the accuracy of the approximation. If for a particular r and a number k of terms of the series used the resulting continued fraction C.F.(r, k) is determined, we shall have

$$\phi_1(x)=\frac{1}{2x}\text{C.F.}(r,k)+\sum_{n=k}^{\infty}(-1)^n\frac{f(-1)f(1)}{(2n+x)f(2n+x-1)f(2n+x+1)}. \tag{14}$$

The error committed by using only the C.F. term is not greater than the first term of the series, that is

$$\frac{f(-1)f(1)}{(2k+x)f(2k+x-1)f(2k+x+1)}.$$

If x be limited to the range that we have been considering, $2\leqq x\leqq 4$, this has its largest value for $x=2$, *viz.*,

$$\frac{f(-1)f(1)}{(2k+2)f(2k+1)f(2k+3)}.$$

For example, if we take the largest values for which we have made the calculations, $r=8$ and $k=3$, the upper bound for the error is

$$\frac{f(-1)f(1)}{8f(7)f(9)}=\frac{1}{8(4r^3+6r^2+8r+3)(2r^4+4r^3+10r^2+8r+3)/9},$$

or about 4 in the eighth decimal place.

6. The series $\phi_2(x)$. This series $\sum_{n=0}^{\infty}(-1)^n(2n+x)^{-2}$ is the negative derivative of $\phi_1(x)$ with respect to x, that is $\phi_2(x) = -\frac{1}{4}\beta'(x/2)$. In this case r odd yields nothing, for then $f(x)$ and $g(x)$ have a common factor which can be removed.

The equations by means of which we calculate the coefficients are now

$$
\begin{aligned}
p_2 - q_2 &= -\, 3E_2,\\
p_4 - q_4 &= -\, 3E_2p_2 - 5E_4,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
p_{r-2} - q_{r-2} &= -\, 3E_2p_{r-4} - 5E_4p_{r-6} - \cdots - (r-1)E_{r-2},\\
p_r &= -\, 3E_2p_{r-2} - 5E_4p_{r-4} - \cdots - (r+1)E_r,\\
0 &= -\, 3E_2p_r - 5E_4p_{r-2} - \cdots - (r+3)E_{r+2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
0 &= -\, 3E_{r-2}p_r - 5E_rp_{r-2} - \cdots - (2r-1)E_{2r-2}.
\end{aligned}
$$

The coefficients of $s^{2r-2}, \cdots, s^2$ in $F_2(s)$ are thus made to vanish, leaving only the constant term. The resulting formula is

$$
(15) \qquad \phi_2(x) = \frac{g(x-1)}{2f(x-1)} + \sum_{n=0}^{\infty}(-1)^n \frac{f(-1)f(1)}{(2n+x)^2 f(2n+x-1)f(2n+x+1)}.
$$

The following table gives the result of this calculation:

r	2	4	6	8	10
p_2	3	22	73	172	335
q_2		19	70	169	332
p_4		41	907	6,838	30,778
q_4			713	6,347	29,798
p_6			1,323	62,604	878,190
q_6				45,963	793,804
p_8				77,841	6,567,221
q_8					4,571,521
p_{10}					7,269,075

7. Rational approximations of $\phi_2(x)$. Since the first term of the series for $x=2$ is $f(-1)f(1)/[2^2f(1)f(3)]$ and $f(3)=(r+1)f(1)$, it again appears that the external term furnishes an approximation to $\phi_2(x)$ in the interval $2 \leq x \leq 4$. The continued fraction of which the successive external terms constitute the successive convergents is

$$
(16) \qquad
\begin{aligned}
2\phi_2(x) &= \frac{1\ |}{|(x-1)^2+3} + \frac{-\,16\ |}{|(x-1)^2+19}\\
&\quad + \frac{-\,256\ |}{|(x-1)^2+51} + \frac{-\,1{,}296\ |}{|(x-1)^2+99} + \cdots.
\end{aligned}
$$

Combining with the external term just the first term of the series that occurs

in (14), we obtain a sequence of rational functions which better approximate $\phi_2(x)$. The general form of these is

$$\frac{1}{x^2} - \frac{g(x+1)}{2f(x+1)},$$

and if, after setting aside the factor $1/2x^2$, they are expressed as continued fractions, they reveal a simple law of formation. We reach the conclusion that $\phi_2(x)$ may be represented approximately by

$$\frac{1}{2x^2}\left[1, \frac{x}{2}, \frac{x}{2}, \frac{x}{4}, \frac{x}{4}, \cdots, \frac{x}{r-2}, \frac{x}{r-2}, \frac{x}{r}, \frac{(x+r)}{r}\right] \tag{17}$$

which suggests an infinite continued fraction, modified when an approximation is desired.

If two terms of the series are used, it is necessary to compute

$$\frac{1}{x^2} - \frac{1}{(x+2)^2} + \frac{g(x+3)}{2f(x+3)}.$$

This yields a new modification of the same infinite continued fraction, that is to say

$$\frac{1}{2x^2}\left[1, \frac{x}{2}, \frac{x}{2}, \cdots, \frac{x}{r}, \frac{(x-r)}{r}, \frac{x+4r+3}{8r+8}, (8r+8)(x+1)\right] \tag{18}$$

is a still better approximation to $\phi_2(x)$.

8. The series $\phi_t(x)$, $t>2$. For the larger values of t the results are not so simple, since there are not enough coefficients $p_2, \cdots$ and $q_2, \cdots$ to enable us to make the numerator independent of n in the new series. It is a question whether the rational approximations obtained are simple enough to be valuable.

9. The non-alternating series. A series of integral powers of the reciprocals of the terms of an arithmetical progression, not involving the alternating factor $(-1)^n$, can be handled in similar fashion if the exponent $t \geqq 2$. Since

$$\sum_{n=0}^{\infty} \frac{1}{(a+nd)^t} = \left(\frac{2}{d}\right)^t \sum_{n=0}^{\infty} \frac{1}{(2n+x)^t},$$

where $x = 2a/d$, the problem is to investigate the function

$$\psi_t(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+x)^t}, \qquad t \geqq 2. \tag{19}$$

From this series we subtract term by term the series

$$\sum_{n=0}^{\infty}\left[\frac{g(2n+x-1)}{2(t-1)f(2n+x-1)} - \frac{g(2n+x+1)}{2(t-1)f(2n+x+1)}\right] = \frac{g(x-1)}{2(t-1)f(x-1)}, \tag{20}$$

in which $f(x)$ and $g(x)$ denote polynomials of degree r and $r-t+1$, respectively, each containing either only odd powers or only even powers of x;

$$\text{(21a)}\qquad f(x) = x^r + p_2x^{r-2} + p_4x^{r-4} + \cdots \begin{cases} + p_r, & \text{if } r \text{ is even}, \\ + p_{r-1}x, & \text{if } r \text{ is odd}, \end{cases}$$

$$\text{(21b)}\qquad g(x) = x^{r-t+1} + q_2x^{r-t-1} + \cdots \begin{cases} + q_{r-t+1}, & \text{if } r-t+1 \text{ is even}, \\ + q_{r-t}x, & \text{if } r-t+1 \text{ is odd}. \end{cases}$$

The coefficients are to be so chosen as to reduce as far as possible the degree of the numerator in n. Replacing $2n+x$ by s we find for this numerator,

$$\text{(22)}\qquad \begin{aligned} F_t(s) = {} & 2(t-1)f(s-1)f(s+1) \\ & - s^t[g(s-1)f(s+1) - g(s+1)f(s-1)], \end{aligned}$$

and the equations obtained by equating to zero the successive coefficients of powers of s are

$$\text{(23)}\qquad \begin{cases} p_2 - q_2 = -\binom{t}{2}D_2, \\ p_4 - q_4 = -\binom{t}{2}D_2p_2 - \binom{t+2}{4}D_4, \\ p_6 - q_6 = -\binom{t}{2}D_2p_4 - \binom{t+2}{4}D_4p_2 - \binom{t+4}{6}D_6, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

These involve, instead of the Eulerian numbers E_ν, a similar class of numbers D_ν, which may be defined by the symbolic equation

$$(D+1)^\nu - (D-1)^\nu = \begin{cases} 0, & \nu > 1, \\ 2, & \nu = 1, \end{cases} \qquad D_0 = 1.$$

When ν is odd $D_\nu = 0$; the first few values of D_ν for ν even are $D_2 = -1/3$, $D_4 = 7/15$, $D_6 = -31/21$, $D_8 = 127/15$, $D_{10} = -2{,}555/33$, $D_{12} = 1{,}414{,}477/1{,}365$, $D_{14} = -57{,}337/3$ [**8**, pp. 27 and 458]. The equations are independent of r and may be thought of as continuing indefinitely. From this point on we treat separately the different cases for t.

10. The series $\psi_2(x)$. If $t=2$, the number of coefficients is just sufficient so that, if they are properly chosen, $F_2(s)$ is independent of s and equal to $2f(-1)f(1)$. We have then

$$\text{(24)}\qquad \psi_2(x) = \frac{g(x-1)}{2f(x-1)} + \sum_{n=0}^{\infty} \frac{f(-1)f(1)}{(2n+x)^2 f(2n+x-1)f(2n+x+1)}.$$

The following table gives the results of calculating the coefficients p and q for a few values of r:

r	2	3	4	5	6
p_2	1/3	7/5	26/7	70/9	155/11
q_2		16/15	71/21	67/9	454/33
p_4			27/35	407/63	329/11
q_4				4,096/945	4,237/165
p_6					375/77

11. Rational approximations to $\psi_2(x)$. It appears that the function $f(x)$ takes on for odd values of x the following values: $f(1)=2^r(r!)^3/(2r)!$, $f(3)=(r^2+r+1)f(1)$, $f(5)=\frac{1}{4}(r^4+2r^3+7r^2+6r+4)f(1)$, $\cdots$. From this we conclude that the external term furnishes an approximation to $\psi_2(x)$. These successive external terms, whose general form is $\frac{1}{2}g(x-1)/f(x-1)$, constitute the successive convergents of an infinite continued fraction. We have therefore a first approximation to $\psi_2(x)$ in the form

$$\frac{1}{2}\left\{\frac{1|}{|x-1}+\frac{1/3|}{|x-1}+\frac{16/15|}{|x-1}+\frac{81/35|}{|x-1}+\frac{256/63|}{|x-1}+\cdots\right\}. \tag{25}$$

A second approximation comes from combining with the external term the first term of the series in the sequence of equations (24). This requires the calculation of

$$\frac{1}{x^2}+\frac{g(x+1)}{2f(x+1)}$$

and its transformation into a continued fraction. The result is

$$\frac{1}{2x^2}\left[x+1,\ \frac{3x}{2},\ \frac{5x}{2\cdot 3},\ \frac{7x}{3\cdot 4},\ \cdots,\ \frac{(2r-1)x}{(r-1)r},\ \frac{x+r}{r}\right], \tag{26}$$

suggesting an infinite continued fraction modified to obtain an approximation. The same infinite continued fraction further modified appears as the result of combining two terms of the series with the external term; the result is

$$\left\{\begin{aligned}&\frac{1}{2x^2}\Bigg[x+1,\ \frac{3x}{2},\ \frac{5x}{2\cdot 3},\ \frac{7x}{3\cdot 4},\ \cdots,\\ &\qquad\qquad \frac{x+3r}{r},\ -\frac{x-(r^2+r-3)}{8},\ -\frac{8(x+r^2+r+1)}{(r^2+r+1)^2}\Bigg].\end{aligned}\right. \tag{27}$$

12. The series $\psi_3(x)$. In applying the same process to the series $\sum_{n=0}^{\infty}(2n+x)^{-3}$, only even values of r can be used, since otherwise $f(x)$ and $g(x)$ would have a common factor which could be divided out. Here again all the coefficients of powers of s in $F_3(s)$ may be made to vanish, leaving just the constant term. The resulting sequence of series is given by

$$\psi_3(x)=\frac{g(x-1)}{4f(x-1)}+\sum_{n=0}^{\infty}\frac{f(-1)f(1)}{(2n+x)^3f(2n+x-1)f(2n+x+1)}. \tag{28}$$

The continued fraction furnished by the external term is

$$\frac{1}{4}\left\{\frac{1\,|}{|(x-1)^2+1} + \frac{-4/3\,|}{|(x-1)^2+5} + \frac{-256/15\,|}{|(x-1)^2+13} + \cdots\right\}, \tag{29}$$

and the modified continued fraction when the first term of the series is combined with the external term is

$$\frac{1}{4x^3}\left\{x+2,\ \frac{x}{2},\ \frac{3x}{2},\ \frac{x}{4},\ \frac{5x}{2\cdot 3},\ \cdots,\ \frac{x}{r},\ \frac{2(x+r)}{r}\right\}. \tag{30}$$

Whether the methods here employed are capable of further application or not, it is believed that the results set forth may throw light on the general problem of the approximate representation of functions by rational functions.

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APPLICATIONS OF THETA FUNCTIONS TO ARITHMETIC

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This paper deals with the application of the Jacobi theta functions to arithmetic. The first applications were made by Jacobi himself and since then by a long list of contributors. Further reference to these is made in the sequel.

The following discussion is written from an elementary point of view, since it is felt that the field deserves a wider acquaintance on the part of those possessing a mathematical background equivalent to the average undergraduate major. Either the results themselves or the proofs of old results are believed to be new.

The four Jacobi theta functions [1] are defined as follows:

$$\theta_0(x) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nx = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nix},$$

$$\theta_1(x) = 2\sum_{n=0}^{\infty} (-1)^n q^{(2n+1)^2/4} \sin (2n+1)x = -i\sum_{n=-\infty}^{\infty} (-1)^n q^{(2n+1)^2/4} e^{(2n+1)ix},$$

$$\theta_2(x) = 2\sum_{n=1}^{\infty} q^{(2n+1)^2/4} \cos (2n+1)x = \sum_{n=-\infty}^{\infty} q^{(2n+1)^2/4} e^{(2n+1)ix},$$

$$\theta_3(x) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nx = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2nix},$$

where $q = e^{\pi i \tau}$, the imaginary part of τ being positive so as to insure convergence. These functions belong to a class named by Hermite [2] doubly periodic of the third kind, but we shall not be concerned here with their detailed properties.

There are in the literature a great many infinite series expansions for various products and quotients of the Jacobi theta functions together with their derivatives. These expansions proceed in powers of q and the coefficients are functions of the divisors of the exponent of q or, more generally, functions of the variables in some form representing the exponent of q. Arithmetical results are then obtained by equating like powers of q in an identity connecting such theta products and quotients.

We start with a simple illustrative example. The arithmetized expansion for $\theta_0'^2(x)/\theta_0^2(x)$, where the prime indicates differentiation, follows [3]:

$$\theta_0'^2(x)/\theta_0^2(x) = 4\sum q^n\left(\sum \delta\right) + 8\sum q^{n'}\left[\sum (\tau - t) \cos 2tx\right]. \tag{1}$$

The outer summation in the first term on the right is an infinite one extending over all positive, even integral values of n. The inner summation occurring as part of the coefficient of different powers of q is finite and extends over all positive, even integral divisors, δ, of n; thus the coefficient of q^6 in the first term is $4(2+6) = 32$. In the second term the n' summation is infinite and is over all positive, integral values; while the inner sum is finite and extends over all positive, integral conjugate divisors t, τ of n', $(n' = t\tau)$, τ being restricted to odd val-

ues only. Thus the coefficient of q^6 is $8\,[(1-6)\cos 12x+(3-2)\cos 4x]$, which reduces to $-40\cos 12x+8\cos 4x$.

From the definition of $\theta_0(x)$ it is evident that $\theta_0'(0)=0$. Placing $x=0$ in (1) we obtain

$$\sum q^n\left(\sum \delta\right) + 2\sum q^{n'}\left[\sum (\tau - t)\right] = 0. \tag{2}$$

Since this is an identity in the q's, the coefficients of the different powers of q must be separately zero. Considering only even values of n' so that $n=n'$, we see that for any even number $\sum\delta+2\sum(\tau-t)=0$, which may be written $\sum(\delta+\tau)+\sum\tau=2\sum t$. This result may be interpreted as follows:

For any even number, the sum of all the divisors plus the sum of the odd divisors is equal to twice the sum of those divisors whose conjugates are odd.

We shall now give an example of the application of theta functions to the problem of the determination of the number of representations of any integer as a sum of squares. An extensive treatment has been given by Glaisher [**4**], and many other writers have written on the problem from various angles [**5**].

The distinctions between a representation, a composition, and a partition of a number as a sum of squares can perhaps best be made clear by an example. There are four representations of 4 as a sum of 2 squares, namely: 2^2+0^2, 0^2+2^2, $(-2)^2+0^2$, and $0^2+(-2)^2$. If positive numbers only had been used, the ways of expressing 4 as a sum of squares would be called compositions and there would be two of them. If the order is also neglected there would be but one way and it would be called a partition. If two or more of the numbers to be squared are alike, both in magnitude and sign, a different representation is not obtained by interchanging the positions of the two like numbers. Although the number of partitions or compositions would seem to be the more significant concept, the results can ordinarily be obtained more directly in terms of representations and are usually so given.

We start from the theta constant identity [**6**]:

$$\theta_2^4 = \theta_0''/\theta_0 - \theta_3''/\theta_3, \tag{3}$$

where $\theta_2=\theta_2(0)$, etc. If we take the exponential definition of $\theta_2(x)$ above, place $x=0$, and $2n+1=m$, an odd number, we have $\theta_2=\sum q^{m^2/4}$. Raising this equation to the fourth power gives

$$\theta_2^4 = \sum\sum\sum\sum q^{(x^2+y^2+z^2+v^2)/4}, \tag{4}$$

and if we write this expression for θ_2^4 as a single summation by collecting the coefficients of like powers of q, thus

$$\theta_2^4 = \sum A_n q^{n/4}, \tag{5}$$

it can be seen that A_n is the number of representations of n as a sum of four odd squares.

The expansions needed for the right side of (3) are [**3**]:

$$\theta_0''(x)/\theta_0(x) = 4\sum q^n(\sum \delta) + 8\sum q^{n'}(\sum \tau \cos 2tx),$$
$$\theta_3''(x)/\theta_3(x) = 4\sum q^n(\sum \delta) + 8\sum q^{n'}[\sum (-1)^t \tau \cos 2tx],$$

where the nature of the summations and values of n, n', δ, t, and τ are the same as in (1) above.

Putting $x=0$ in these expansions and substituting the results along with (5) in (3) gives

$$\sum A_n q^{n/4} = 8\sum q^n[\sum \tau(1 - (-1)^t)]. \tag{6}$$

Note that if we are to equate like powers of q on the two sides of this equation, the n on the left must be taken equal to four times the value of the n on the right. Keeping this in mind and considering odd values of n only, so that n, t, τ, are all odd in $n = t\tau$ on the right, we may deduce from (6) a classical theorem of Jacobi [7]:

"The number of representations of the quadruple of any odd number, n, as a sum of four odd squares is equal to 16 times the sum of the divisors of n."

If n had been taken equal to an even number in (6), we could have concluded that there are no representations of the quadruple of any even number as the sum of 4 odd squares, a result which is otherwise fairly obvious.

If we now raise both sides of (6) to the mth power and rearrange by collecting the coefficients of like powers of q, there results

$$\sum B(4m, n)q^{n/4} = 8^m \sum q^N \sum \tau_1\tau_2 \cdots \tau_m[1-(-1)^{t_1}][1-(-1)^{t_2}] \cdots [1-(-1)^{t_m}].$$

As can perhaps best be seen by referring to (4) and (5), $B(4m, n)$ is the number of representations of n as a sum of $4m$ odd squares. The inner summation on the right is the product of the original m finite summations and is over all combinations of t's and τ's which satisfy, for any particular N, the equation

$$N = n_1 + n_2 + \cdots + n_m = t_1\tau_1 + t_2\tau_2 + \cdots + t_m\tau_m.$$

Equating like powers of q, and remembering that $n = 4N$, we get

$$B(4m, 4N) = 8\sum \tau_1\tau_2 \cdots \tau_m[1 - (-1)^{t_1}][1 - (-1)^{t_2}] \cdots [1 - (-1)^{t_m}].$$

If either N or m is odd while the other is even, $B(4m, 4N)$ is zero. This can be seen since the τ's are odd by assumption and all the t's must be odd in order for $B(4m, 4N)$ to be different from zero. When m and N are either both odd or even, however, there results

$$B(4m, 4N) = 16^m \sum \tau_1\tau_2 \cdots \tau_m, \tag{7}$$

where in

$$N = t_1\tau_1 + t_2\tau_2 + \cdots + t_m\tau_m$$

all t's and τ's are odd.

It is possible by means of certain other theta function identities to deduce from (7) the classical theorem on the representation of the quadruple of any even number as the sum of eight odd squares [8], and certain other results for

12 and 16 odd squares, but we shall not take space here to develop this line of inquiry further.

Instead we shall take up the method of paraphrase [**9**] which adds a great deal to the generality of any results obtained. The functions of the divisors of n occurring as coefficients of q^n in the arithmetized series developments usually contain sine or cosine terms as in the above examples. Thus theorems concerning trigonometric functions are obtained by equating like powers of q. But, as is well known from the theory of Fourier series, the majority of analytic functions are, within certain limitations, representable by trigonometric series. Hence it is frequently possible by giving suitable particular values to the variables in these theorems and then summing up, to extend a theorem established for trigonometric functions to all functions which are representable by convergent series of the trigonometric function for which the theorem was first established. Thus, under certain conditions, a result concerning cosine terms can be extended to even functions, and a theorem concerning sine terms to odd functions.

An example will make these statements more definite. In a list of arithmetized expansions given by Bell [**10**] occurs

$$\theta_0'(x)/\theta_0(x) = 4\sum q^n(\sum \sin 2tx), \tag{8}$$

where in $n=t\tau$, t is an arbitrary positive integer, and τ is an odd positive integer. Throughout the sequel, τ will always represent a positive odd integer, and d, δ, t arbitrary positive integers unless otherwise indicated. We shall not point out where the latter are necessarily restricted, *e.g.*, t when n is odd in $n=t\tau$.

It is obvious that if we square (8) it may be equated to (1) above, thus obtaining

$$4\sum q^n(\sum \delta) + 8\sum q^{n'}\sum (\tau - t)\cos 2tx = 16\sum q^N(\sum \sin 2t_1x \sin 2t_2x),$$

where $N=n_1+n_2=t_1\tau_1+t_2\tau_2$. From trigonometry the right side can be written $8\sum q^N[\sum \cos 2(t_1-t_2)x-\cos 2(t_1+t_2)x]$. Equating like powers of q, there results

$$\sum_n \delta + 2\sum_{n'} (\tau' - t')\cos 2t'x = 2\sum_N [\cos 2(t_1 - t_2)x - \cos 2(t_1 + t_2)x], \tag{9}$$

where, to sum up, $n=d\delta$, $n'=t'\tau'$, $N=t_1\tau_1+t_2\tau_2$, and δ is given.

Let $f(y)$ be an even analytic function, well defined and finite everywhere between the limits $-h$ and h. Then $f(y)$ can be expanded in a Fourier series as follows:

$$f(y) = A_0/2 + A_1\cos \pi y/h + A_2\cos 2\pi y/h + \cdots + A_m\cos m\pi y/h + \cdots. \tag{10}$$

Next put $y=0$, $2t'$, $2(t_1-t_2)$, $2(t_1+t_2)$ successively in (10), and in (9) put x successively equal to π/h, $2\pi/h$, $\cdots$, $m\pi/h$, $\cdots$. Now multiply both sides of (9) by A_1 for $x=\pi/h$, by A_2 for $x=2\pi/h$, and in general by A_m for $x=m\pi/h$. If we add the resulting equalities and take into consideration the results of the substitution in (10), we have

$$\sum_n \delta f(0) + 2\sum_{n'} (\tau' - t')f(2t') = 2\sum_N f(2t_1 - 2t_2) - f(2t_1 + 2t_2). \tag{11}$$

(11) is called the paraphrase of (9) and is a much more general relationship. For a cosine series the paraphrase substitution is

$$\sum (a_i + b_i \cos c_i y) = \sum a_i f(0) + b_i f(c_i). \tag{12}$$

For an identity containing sine terms only it can be shown that the proper substitution is

$$\sum (a_i \sin b_i y) = \sum a_i F(b_i), \tag{13}$$

where $F(y)$ is an odd function.

If we place $f(y)$ equal to a constant in (11) we get the same result that was obtained from (2). If we place $f(y)=y^2$, we obtain

$$\sum_{n'} (t' - \tau')t'^2 = 4\sum_N t_1 t_2,$$

a result connecting the divisors of n' with two of the four variables in the form $N=t_1\tau_1+t_2\tau_2$. A check of this result for $N=n'=4$ gives $48=48$.

The even and odd functions considered above do not have to be continuous. It can be shown that it is only necessary for them to have definite, finite values for the integral values of the independent variable under consideration. For example, if in (11) we place $f(0)=1$, $f(y)=0$ for all other integral values of y, we may conclude that the sum of the even divisors of any number, n, is equal to twice the number of times in which $t_1=t_2$ in the compositions of n by means of $t_1\tau_1+t_2\tau_2$.

We give one further example for which we need the following expansion obtained by Basoco [11]:

$$\begin{aligned}\frac{\theta_1'^2\theta_0^2(x+y)}{\theta_0^2(x)\cdot\theta_1^2(y)} &= \frac{2\theta_1'(y)}{\theta_1(y)}\left[\csc 2y + 4\sum q^n\left(\sum \sin 2(tx+\tau y)\right)\right] \\ &\quad - 8\sum q^n \sum t\cos 2(tx+\tau y).\end{aligned} \tag{14}$$

In the list of expansions given by Bell [5] occur:

$$\theta_1'^2\theta_0^2/\theta_0^2(x)\theta_1^2(x) = \csc^2 x - 8\sum q^{n_1}\left[\sum d_1 \cos 2(d_1-\delta_1)x\right], \tag{15}$$

$$\theta_1'(y)/\theta_1(y) = \cos y + 4\sum q^{n'}\left(\sum \sin 2d'y\right), \tag{16}$$

where δ' is even in (16) in $n'=d'\delta'$.

If we set $y=-x$ in (14) it is evident that it is identically equal to (15) and the two right-hand members may be equated. Terms free of q must be equal in such an identity. If we drop these terms out, multiply through by $\sin 2x$, use trigonometric formulas, and paraphrase by means of (13), there results after considerable detailed labor

$$\begin{aligned}\sum_{n'} 2F(2d') - \sum_n \{2F(2t-2\tau) + F(2t-2\tau+2) + F(2t-2\tau-2) \\ + tF(2t-2\tau+2) - tF(2t-2\tau-2)\} \\ + \sum_{n_1} \{d_1F(2d_1-2\delta_1+2) - d_1F(2d_1-2\delta_1-2)\} = R,\end{aligned}$$

where

$$R = 2\sum_{N} \{F(2t - 2\tau - 2d' + 2) - F(2t - 2\tau - 2d' - 2) \\ - F(2t - 2\tau + 2d' + 2) + F(2t - 2\tau + 2d' - 2)\},$$

and where $n' = d'\delta'$, δ' even, $n = t\tau$, $n_1 = d_1\delta_1$, $N = t\tau + d'\delta'$.

If we set $F(y) = y$, we obtain

$$\sum_{n'} d' - \sum_{n} (3t - 2\tau) + \sum_{n} d = 0,$$

a result which may be interpreted as follows:

For any even number, *the sum of the divisors whose conjugates are even plus the sum of all divisors is equal to three times the sum of all divisors whose conjugates are odd minus twice the sum of the odd divisors.*

These examples deal only with a few of the more elementary applications to number theory. Further applications may be found, among other places, in the works of Hermite, Jacobi, Legendre, Kronecker, Liouville, Humbert, and Glaisher. Particular attention is called to numerous applications made by E. T. Bell in papers occurring in many journals during the past twenty years, and to a book on the subject by Nazimoff [**12**].

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ON CONGRUENCES INVOLVING SUMS OF PRODUCTS OF BINOMIAL COEFFICIENTS

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In some investigations the writer has been carrying out recently it has been found convenient to employ certain lemmas involving congruences of the type indicated in the title. These are generalizations of a result given by Glaisher* who found that

$$\text{(A)} \qquad \binom{a}{i} \equiv \binom{d}{i} + \binom{d}{i+(p-1)} + \binom{d}{i+2(p-1)} + \cdots, \pmod{p},$$

where $d \equiv a \pmod{\overline{p-1}}$; $0 < a \leqq p-1$; p is prime, $0 < i \leqq p-1$; and $\binom{v}{w} = 0$, when $w > v$.

We give here only the simplest of types of congruences we shall need elsewhere. There are extensive generalizations which we shall indicate how to obtain, without writing out the results explicitly, as some of the latter are quite complicated.

Let k and m be arbitrary integers and p a prime; then

$$\text{(1)} \qquad (m+k)^c[(m+k)^{p-1}-1]^r \equiv 0 \pmod{p^r}, \qquad c \geqq r,$$

since if $(m+k)$ is prime to p, then $(m+k)^{p-1} \equiv 1 \pmod{p}$; and if $m+k \equiv 0 \pmod{p}$ then $(m+k)^c \equiv 0 \pmod{p^r}$, since $c \geqq r$. Expanding (1), we have

$$\text{(1a)} \qquad \sum_{h=0}^{r} (-1)^h \binom{r}{h} (m+k)^{h(p-1)+c} \equiv 0 \pmod{p^r}.$$

If we select k so that

$$\text{(2)} \qquad k^{p-1} \equiv 1 \pmod{p^r},$$

we may employ this in (1a), collect coefficients of powers of k, and obtain

$$\text{(3)} \qquad \sum_{h=0}^{r} \sum_{\alpha=0}^{p-2} \sum_{l=0,1,\cdots} (-1)^h \binom{r}{h} \binom{(p-1)h+c}{(p-1)l+\alpha} k^{\beta} m^{l(p-1)+\alpha} \equiv 0 \pmod{p^r},$$

where β is the integer in the set $0, 1, \cdots, p-2$, such that

$$\beta \equiv c - \alpha \pmod{\overline{p-1}}.$$

Hence as α ranges over the integers $0, 1, \cdots, p-2$, then β ranges over the same set in some order. Consequently (3) may be written in the form

$$\text{(4)} \qquad A_0 + A_1 k + \cdots + A_{p-2} k^{p-2} \equiv 0 \pmod{p^r}.$$

We may now select for k in (2) an integer k_a which also has the property that $k_a \equiv a \pmod{p}$; $0 < a < p$, by known methods. In (4) set $k = k_1, k_2, \cdots, k_{p-1}$ in

* Quarterly Journal of Mathematics, vol. 30, 1899, pp. 361–366. This was referred to by Dickson as a very interesting result.

turn, then the determinant of the coefficients of the A's in the resulting congruences is the alternant

$$\begin{vmatrix} 1 & k_1 & \cdots & k_1^{p-2} \\ 1 & k_2 & \cdots & k_2^{p-2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 1 & k_{p-1} & \cdots & k_{p-1}^{p-2} \end{vmatrix} = \prod_{i,j} (k_i - k_j), \qquad i \neq j,$$

which is prime to p. Hence

$$A_0 \equiv A_1 \equiv \cdots \equiv A_{p-2} \equiv 0 \pmod{p^r};$$

or, using the values of the A's as obtained from (3) and dividing by m^α, assuming m prime to p, we find

$$\sum_{h=0}^{r} \sum_{l=0,1,\cdots} m^{l(p-1)}(-1)^h \binom{r}{h} \binom{(p-1)h+c}{(p-1)l+\alpha} \equiv 0 \pmod{p^r}. \tag{5}$$

Now select m so that

$$m^{p-1} = m_g^{p-1} \equiv 1 + gp \pmod{p^r},$$

where g is a given integer, which is always possible; then

$$m_g^{(p-1)l} = (1 + gp)^l. \tag{6}$$

Employing the expansion of (6) in (5) the result may be written in the form

$$B_0 + B_1 gp + \cdots + B_{p-2}(gp)^{p-2} \equiv 0 \pmod{p^r}.$$

Now assume $g = 1, 2, \cdots, p-2$, in turn; the above gives a set of congruences in which the determinant of the coefficients of $B_0, B_1 p, \cdots, B_{p-2}p^{p-2}$ is an alternant formed by powers of the g's which is prime to p since each $g<p$. Hence

$$B_i p^i \equiv 0 \pmod{p^r}, \qquad i = 0, 1, \cdots, p-2;$$

or using the values of the B's from (5) we have for $i<p-1$,

$$\sum_{h=0}^{r} \sum_{l=0,1,\cdots} (-1)^h p^i \binom{l}{i} \binom{r}{h} \binom{(p-1)h+c}{(p-1)l+\alpha} \equiv 0 \pmod{p^r},$$

which gives in particular for $i=0$,

$$\sum_{h=0}^{r} \sum_{l=0,1,\cdots} (-1)^h \binom{r}{h} \binom{(p-1)h+c}{(p-1)l+\alpha} \equiv 0 \pmod{p^r}.$$

Hence we have the

THEOREM. *Let p be prime, α any integer in the set $0, 1, \cdots, p-2$; $c \geqq r$; $i<p-1$; $r>i$; then*

$$\sum_{h=0}^{r} \sum_{l=0,1,\cdots} (-1)^h \binom{r}{h} \binom{(p-1)h+c}{(p-1)l+\alpha} \binom{l}{i} \equiv 0 \pmod{p^{r-i}}. \tag{7}$$

We now consider some generalizations. For example, instead of using (1) we can employ

$$(m+k)^c[(m+k)^{d(p-1)}-1]^r \equiv 0 \ [\mathrm{mod}\ (p^{rs}, p^c)],$$

where $d=p^{s-1}d_1$; d_1 prime to p, where (x, y) is the greatest common divisor of x and y. This gives

$$\sum_{h=0}^{r} (-1)^h \binom{r}{h} (m+k)^{hd(p-1)+c} \equiv 0 \ [\mathrm{mod}\ (p^{rs}, p^c)].$$

We may then select k so that

$$k^{p-1} \equiv 1 \pmod{p^{rs}},$$

which is an extension of (2), but instead we might select k so that

$$k^{p-1} \equiv 1 + jp \pmod{p^{rs}},$$

where j is an arbitrary integer, and proceed as we did in the treatment following (6) with k in place of m. Then later we may treat m as we did in (6) and the relation which precedes. This will give in lieu of (7) a congruence which involves in general, in each term, the product of four binomial coefficients. For the particular case when $r=s=1$ we obtain Glaisher's relation (A).

A further generalization is obtained by considering

$$(m_1+m_2+\cdots+m_a)^c[(m_1+m_2+\cdots+m_a)^{d(p-1)}-1]^r \equiv 0 \ [\mathrm{mod}\ (p^{rs}, p^c)],$$

and expanding the left-hand member by the multinomial and binomial expansions. For $r=s=1$ this reduces to a result concerning multinomial coefficients proved by Dickson.* In another paper† the writer obtained the result that

$$\binom{a}{i} \equiv \binom{d}{i} + \binom{d}{i+p^n-1} + \binom{d}{i+2(p^n-1)} + \cdots$$

modulo p, when $d \equiv a \pmod{p^n-1}$; $0<a\leqq p^n-1$; $0<i\leqq p^n-1$; which is a different type of generalization of (1). This was proved by the use of Galois (finite) field theory. To obtain results of this kind modulo p^α apparently it is necessary to employ finite rings instead of finite fields. We may start with the relatively simple type of finite ring formed by the residue classes with respect to the modulus $\mathfrak{p}^\alpha$ where $\mathfrak{p}$ is a prime ideal divisor of (p) in an algebraic field K, note that

$$(\rho_1+\rho_2+\cdots+\rho_a)^c[(\rho_1+\rho_2+\cdots+\rho_a)^{d(p^n-1)}-1]^r \equiv 0 \ [\mathrm{mod}\ (\mathfrak{p}^{rs}, \mathfrak{p}^c)]$$

where now $d=p^{n(s-1)}d_1$; $\rho_1, \rho_2, \cdots, \rho_a$ are arbitrary integers in K, and then proceed as in the treatment of (1a).

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* Quarterly Journal of Mathematics, vol. 33, 1902, pp. 381–384.

† Annals of Mathematics, vol. 28, 1927, p. 332.

QUESTIONS, DISCUSSIONS, AND NOTES

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REMARKS ON PENCILS OF CIRCLES AND NETS OF SPHERES

N. A. COURT, University of Oklahoma

1. If the points P and P', Q and Q', R and R' divide the sides BC, CA, AB of the triangle ABC externally and internally in ratios such that the product of the three ratios is unity, the points P, Q, R are collinear, by Menelaus's theorem, and the points of division considered are three pairs of opposite vertices of a complete quadrilateral of which ABC is the diagonal triangle.

It is therefore possible to describe three circles (A), (B), (C) having A, B, C for centers so that the centers of similitude of these three circles shall coincide with the points P, P'; Q, Q'; R, R', as was shown in this MONTHLY, 1929, p. 56, problem 3295.

Thus the circles (PP'), (QQ'), (RR') described on PP', QQ', RR' as diameters are the circles of similitude of the circles (A), (B), (C), and the properties pointed out by J. H. Weaver (this MONTHLY, 1938, pp. 17 ff.) become immediately apparent.

When the points P, P', $\cdots$ are the traces of the bisectors of the angles of ABC, the radii of the circles (A), (B), (C) are proportional to the respective altitudes of the triangle ABC (*The Tôhoku Mathematical Journal*, vol. 39, Part 2, June, 1934, p. 264, art. 1).

2. If the points P and P', Q and Q', R and R', U and U', V and V', W and W' divide the edges BC, CA, AB, DA, DB, DC of the tetrahedron $DABC$ externally and internally in ratios such that the points P, Q, R, U, V, W are coplanar, the twelve points of division form a desmic system.*

It is possible to describe four spheres (A), (B), (C), (D) having A, B, C, D for centers such that the plane $PQRUVW$ shall be a plane of similitude of these spheres.† Thus the spheres (PP'), $\cdots$, (WW') described on PP', $\cdots$, WW' as diameters are the six spheres of similitude of the spheres (A), $\cdots$, (D) taken in pairs. These six spheres therefore form a coaxal net, are orthogonal to the circumsphere of the tetrahedron $ABCD$, *etc.*‡

When the points P, P', $\cdots$, W, W' are the traces of the bisecting planes of the dihedral angle of $ABCD$ upon the respectively opposite edges, the radii of the spheres (A), $\cdots$, (D) are proportional to the respective altitudes of the tetrahedron $ABCD$.§

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* Nathan Altshiller-Court, Modern Pure Solid Geometry, Macmillan, New York, 1935, p. 237, art. 728.

† Ibid., p. 206, art. 642. The Mathematics Student, vol. 3, 1935, p. 2.

‡ Ibid., p. 203, art. 634. § Ibid., p. 252, arts. 768 ff.

A SIMPLE APPROXIMATION FOR π

M. G. GABA, University of Nebraska

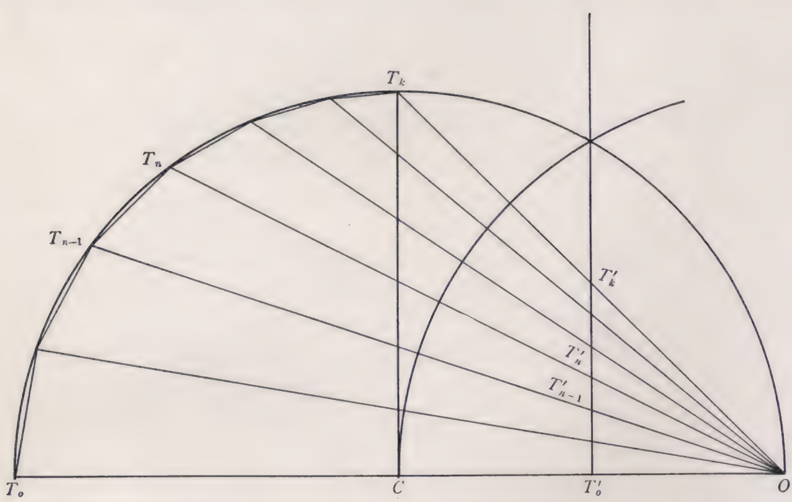
John Wallis is given credit for being the first mathematician to express π as the limit of a sequence of *rational* numbers. The earlier geometrical methods of inscribing and circumscribing regular polygons in or about a circle led to approximations involving successive extractions of square roots. It has been generally believed that the methods of the "Newer Analysis" introduced during the latter half of the seventeenth century are necessary in order to evaluate π as a limit of rational numbers. That this is not so will be shown by proving by elementary means that

$$\pi = \lim_{k\to\infty} \pi_k,$$

where

$$\pi_k = \frac{4k-2}{4k^2+1} + \frac{16k^3}{4k^2+1}\sum_{n=1}^{n=k}\frac{1}{n^2+k^2}.$$

Let T_0T_k be a quadrant of arc of a unit circle whose center is C and T_0O a



diameter. The inverse of this quadrant with respect to the unit circle whose center is at O will be the line segment $T'_0T'_k$. Obviously $OT'_0 = T'_0T'_k = \frac{1}{2}$. Divide $T'_0T'_k$ into k equal parts and call the points of division $T'_1, T'_2, \cdots, T'_{k-1}$. Let the inverses of these points on the arc T_0T_k be $T_1, T_2, T_3, \cdots, T_{k-1}$.

The areas of triangles $T'_{n-1}OT'_n$ are all equal to $1/(8k)$. If we call this area Δ and that of the triangle $T_{n-1}OT_n$ is called Δ_n, then

$$\frac{\Delta_n}{\Delta} = \frac{OT_{n-1}\cdot OT_n}{OT'_{n-1}\cdot OT'_n}.$$

But since

$$(OT_n')^2 = \left(\frac{n}{2k}\right)^2 + \frac{1}{4}$$

and OT_n' and OT_n are reciprocals of each other, we have

$$\frac{\Delta_n}{\Delta} = \frac{1}{(OT_{n-1}')^2(OT_n')^2} = \frac{1}{\left[\left(\frac{n-1}{2k}\right)^2 + \frac{1}{4}\right]\left[\left(\frac{n}{2k}\right)^2 + \frac{1}{4}\right]},$$

$$\Delta_n = \frac{2k^3}{[(n-1)^2 + k^2][n^2 + k^2]}.$$

The area of the polygon $OT_0T_1T_2 \cdots T_k$ is

$$\sum_{n=1}^{n=k} \Delta_n = 2k^3 \sum_{n=1}^{n=k} \frac{1}{[(n-1)^2 + k^2][n^2 + k^2]}.$$

Since

$$\frac{1}{[(n-1)^2 + k^2][n^2 + k^2]} = \frac{1}{4k^2 + 1}\left[\frac{-2n+3}{(n-1)^2 + k^2} + \frac{2n+1}{n^2 + k^2}\right],$$

and

$$\sum_{n=1}^{n=k} \frac{-2n+3}{(n-1)^2 + k^2} = \frac{1+2k}{2k^2} + \sum_{n=1}^{n=k} \frac{-2n+1}{n^2 + k^2},$$

we have

$$\sum_{n=1}^{n=k} \Delta_n = \frac{4k^3}{4k^2+1} \sum_{n=1}^{n=k} \frac{1}{n^2 + k^2} + \frac{k(1+2k)}{4k^2+1}.$$

The area of the polygon $CT_0T_1T_2 \cdots T_k$ is that of the polygon $OT_0T_1T_2 \cdots T_k$ less the area of the triangle OCT_k. If we let $\pi_k/4$ be the area of $CT_0T_1 \cdots T_k$, then

$$\frac{\pi_k}{4} = \sum_{n=1}^{n=k} \Delta_n - \frac{1}{2} = \frac{4k^3}{4k^2+1} \sum_{n=1}^{n=k} \frac{1}{n^2 + k^2} + \frac{k - \frac{1}{2}}{4k^2+1}.$$

The area of the quadrant CT_0T_k is $\pi/4$. But we see that as k tends to infinity the area of the polygon $CT_0T_1 \cdots T_k$ approaches the area of the quadrant CT_0T_k as a limit, and therefore $\pi = \lim_{k\to\infty}\pi_k$, as we proposed to prove.

Computation shows that $\pi_{70} = 3.1413$; the sequence is therefore seen to be not very rapidly convergent.

Note by the Editor. It readily follows from the foregoing short article that

$$\text{(a)} \qquad \frac{\pi}{4} = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{k}{n^2 + k^2},$$

which formula is seen to be closely related to

(b) $$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}.$$

ITERATED AND DOUBLE LIMITS

ARTHUR SARD, Queens College

1. *Let $s(x, y)$ be a function with the property that the limits*

(1) $$s_1(y) = \lim_{x\to\infty} s(x, y), \qquad s_2(x) = \lim_{y\to\infty} s(x, y)$$

exist for all positive y and x, respectively. We shall be concerned with the iterated and double limits of $s(x, y)$:

$$s_{12} = \lim_{y\to\infty} s_1(y), \qquad s_{21} = \lim_{x\to\infty} s_2(x), \qquad s = \lim_{(x,y)\to(\infty,\infty)} s(x, y).$$

It is evident that *if the double limit, s, exists, the iterated limits, s_{12} and s_{21}, exist; and all three limits are equal.* And it is well known that a sufficient condition for the existence of s is that either one of the limits (1) be uniformly approached over the entire range of the remaining variable.* We shall prove

THEOREM A. *A necessary and sufficient condition that the double limit, s, exists is the following: Given any sequence $\{y_j\}$ which tends to infinity, then $s(x, y_j) \to s_1(y_j)$ uniformly over $\{y_j\}$ as x tends to infinity.*

We may express the condition briefly thus: $s(x, y)\to s_1(y)$ uniformly over sequences $\{y\}$.

A. Pringsheim has made an extended investigation of the relation between the iterated limits and the double limit of a double sequence.† He introduces a generalization of the concept of uniform convergence which permits him to dispense with the assumption that the limits (1) exist. However, none of his results has the simplicity of Theorem A. Theorem A shows exactly the rôle played by uniform convergence in our problem.

2. To prove Theorem A we first consider

THEOREM B. *Let $\{x_i\}$, $\{y_j\}$ be two sequences, each of which tends to infinity. Then the double limit $s = \lim_{(i,j)\to(\infty,\infty)} s(x_i, y_j)$ exists if and only if $s(x_i, y_j \to s_1(y_j)$ uniformly.*

Proof: The sufficiency of the condition is well known.

To prove the necessity, observe that if s exists, s_{12} also exists and equals s. Let $\epsilon > 0$ be given. Then there is an integer, N, such that

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* See, for example, W. F. Osgood, Funktionentheorie, Leipzig, 1928, I, pp. 617–623. The ideas of the present paper occurred to the author after study of this portion of the Funktionentheorie.

† A. Pringsheim, Zur Theorie der zweifach unendlichen Zahlenfolgen, Mathematische Annalen, vol. 53, 1900, pp. 289–321; and Vorlesungen über Zahlen und Funktionenlehre, vol. I, part 1, pp. 269–292.

$$|s - s_1(y_j)| < \epsilon/2 \text{ for all } j > N;$$

and an integer, N', such that

$$|s(x_i, y_j) - s| < \epsilon/2 \text{ for all } i, j > N'.$$

Hence

$$|s(x_i, y_j) - s_1(y_j)| < \epsilon \qquad \text{for all } j > N, N'; i > N'.$$

On the other hand, since $s_1(y_j)$ exists, there is an integer, $M(j)$, dependent on j, such that

$$|s(x_i, y_j) - s_1(y_j)| < \epsilon \qquad \text{for all } i > M(j).$$

Now let Q be the maximum of the finite set of numbers:

$$M(1), M(2), \cdots, M(N), \cdots, M(N'), N'.$$

Then the last two inequalities imply that

$$|s(x_i, y_j) - s_1(y_j)| < \epsilon \qquad \text{for } i > Q \text{ and for all } j;$$

that is, $s(x_i, y_j)$ approaches $s_1(y_j)$ uniformly, as was to be shown.

Theorem A is an immediate consequence of Theorem B and the following lemma, the proof of which is left to the reader.

LEMMA. *The limit* $\lim_{x\to\infty} s(x, y_j)$ *is uniform if and only if, for each sequence* $\{x_i\}$ *that tends to infinity, the limit* $\lim_{i\to\infty} s(x_i, y_j)$ *is uniform.*

3. With obvious changes Theorems A and B apply equally well to the case in which x or y or both approach finite limits. The theorems apply also to the case in which x and y are points in any two topological spaces. This permits a certain elegance to be introduced in some practical applications. For example, in considering an iterated integral that is improper at both limits of integration of both variables, one need not resort to the common device of replacing the integral by the sum of other integrals, each of which is improper at only one limit of integration.

Uniform convergence over sequences $\{y\}$ is not equivalent to uniform convergence over a y interval, $a \leqq y < \infty$. Thus consider the function $s(x, y)$ defined for $x > 1$, $y > 1$ as follows:

$$\begin{cases} s(x, y) = 0 & \text{if } \; y - [y] \geqq 1/x,^* \\ s(x, y) = \sin\{\pi x(y - [y])\}/y & \text{if } \; y - [y] < 1/x. \end{cases}$$

For this function, the limits $s_1(y)$, $s_2(x)$, s exist. Hence $s(x, y) \to s_1(y)$ uniformly over sequences $\{y\}$. On the other hand it is clear that the convergence is not uniform over any y interval of length $\geqq 1$.

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* $[y]$ is the largest integer $\leqq y$.

RECENT PUBLICATIONS

EDITED BY TOMLINSON FORT, Lehigh University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

On the Statistical Theory of Errors. By W. E. Deming and R. T. Birge. Washington, D. C., The Graduate School of the United States Department of Agriculture, 1937. 4+42 pages. $.35.

This work presents under a single cover a selection of topics which formerly were rather scattered in the mathematical literature.

A complete table of the symbolism employed in the pamphlet is followed by a brief introduction explaining in a general way the problems considered later. The frequency curve for the parent population is assumed to be normal. The authors justify this choice and answer some objections that have been advanced in the past for such a choice. Next, the distribution laws for $u=$(deviation of mean of sample from mean of parent population), $s=$(standard deviation of sample), and $z=u/s$ are derived. Following this is a section devoted to tests concerning hypotheses that a given sample was drawn from a normally distributed parent population with assumed parameters. These tests are five in number and are clarified with reference to the u, s frequency surface. The notion of fiducially related values of σ and s then follows. Consideration is now given to the problem of estimation from viewpoints of (1) the method of maximum likelihood, (2) empirical estimates, and (3) the posterior method.

Throughout the pamphlet the authors take considerable care in presenting the assumptions underlying the various tests and methods of estimation. Cautionary measures to be taken in applying the various tests are presented. The use of tables for calculation is described at some length and solved problems illustrating the applications of the theory are exhibited.

The typographical arrangement in a few instances leaves something to be desired in that at these places the content of the pamphlet is difficult to follow.

The booklet is especially recommended to those who desire a lucid presentation of the recent statistical viewpoint concerning the theory of errors.

F. S. BEALE

Lectures and Conferences on Mathematical Statistics. By J. Neyman. (Revised and supplemented by J. Neyman with the editorial assistance of W. E. Deming.) Washington, D. C., The Graduate School of the United States Department of Agriculture, 1938. 8+163 pages. $1.25.

This book consists of a report on a series of three lectures and six conferences dealing with various phases of mathematical statistics held at Washington, D. C., in April 1937.

Lecture I deals with the theory of probability in general. The author defines first a fundamental probability set and then proceeds to a definition of probability for each of two cases, (1) when the fundamental probability set is finite and (2) when this set is infinite. A characteristic feature of this definition consists in its expressed avoidance of the term "equally probable cases."

The second lecture contains a statement of the empirical law of big numbers. A large part of this lecture is devoted to a discussion of the construction of a bridge between the theory of probability and certain fields of application. This bridge is made possible through care in constructing a suitable mathematical model together with equal care in performing suitable experiments.

The final lecture, entitled "The testing of statistical hypotheses," is one of the most interesting in the entire book. The question of the retention or rejection of hypotheses together with tests for so doing are considered at some length. Former practices along these lines are mentioned in some detail and their general insufficiency exposed. Consideration is given in the lecture to possibilities of (1) rejecting a true hypothesis and (2) not rejecting a wrong hypothesis. Toward the end of the chapter the procedure of considering a set of n observed values of a property as defining a point in n-way space is introduced—a procedure which has been very helpful in recent times in the solution of certain statistical problems.

The first two conferences dealt mainly with certain applications of statistics to questions relating to agriculture. The third conference treated statistics as applied to social and economic research. It is devoted mainly to the question of procedure in selecting a suitable sample from a heterogeneous population. The main idea appears to be that such a population should be partitioned into homogeneous subdivisions or strata and the total sample should be selected from the members of these strata. A mathematical justification of this procedure is included in the chapter.

The fourth conference concerned itself with analysis of time series. The prediction of economic processes can be attacked in either one of two ways. First, the empirical method which simply concerns itself with final economic results and second, the *a priori* method which concerns itself with the inner workings of the economic machinery. Although the former of the two methods has been mainly employed in the past, the lecturer leans to the thesis that the *a priori* method has been too long neglected. He seems to feel that the best results probably are obtainable through a more prevalent use of the second method.

The final two conferences considered the problem of statistical estimation. Given the form of the distribution law of some property, how can we use a set of observed values of this property to estimate the unknown parameters of the distribution law. The principle of insufficient reason employed in the past was mentioned and a discussion of the "best unbiased" estimate which has been treated extensively by Markoff. Finally, the lecturer launched into a discussion of the problem of estimation from the standpoint of upper and lower estimates of a parameter, and the book closes with a general discussion of the theory of confidence intervals.

The lectures were delivered to an audience which already possessed considerable knowledge in mathematical statistics. It is to such a class of readers that the book has its greatest appeal. Frequent references are given throughout the book for more detailed study. A notable feature of the conferences consists in the inclusion of various questions put to the lecturer by his audience followed by the lecturer's response. The book is especially recommended for those who wish a general statement of modern mathematical statistics problems together with an up-to-date account of the procedure of attack on those problems employed mainly by Pearson and Neyman.

F. S. Beale

Computation and Trigonometry. By H. J. Gay. New York, The Macmillan Company, 1938. 7+231 pp. \$1.90.

This trigonometry, which I have used as a supplementary text for six weeks, is an unusually well written textbook. One of its many good points is the chapter having to do with accuracy, a subject many texts neglect. The various topics are clearly presented in an interesting manner. Approximately five per cent of the large number of well selected problems have been solved. There is a high degree of accuracy in the answers. Many of the problems provoked lively and interesting discussions in the class. The chapter, Solid Geometry and Trignometry, is something new in a trigonometry, and in many classes it should help correct a decided weakness. Other special features are a chapter on spherical trigonometry with applications to navigation and astronomy, and a chapter on the slide rule, both of which are well done.

William Beverley

First Year College Mathematics. By V. H. Wells. New York, D. Van Nostrand Company, 1937. Part I: *Trigonometry*. 7+133 pages. \$1.25. Part II. *Mathematical Analysis*. 9+276 pages. \$2.75.

These two books constitute a modification of the idea of unified mathematics for a first year college course. It is an interesting departure. We quote from the preface of Part II: "The author believes that the subject of trigonometry does not lend itself to a correlation with the subjects of algebra, analytical geometry, and elementary calculus, but that the latter subjects are closely related and are readily studied simultaneously." Acting on this belief the author has made trigonometry the subject matter of Part I. In this part there is nothing strikingly original in the selection and arrangement of material. Nevertheless, the presentation is fresh and stimulating. To quote from the preface: "One of the author's main objectives has been to write a text requiring as little memorization as possible, but rather, to develop an understanding of the underlying and fundamental principles of the subject." In the reviewer's opinion this is a commendable objective, and the author has reached it to a very gratifying degree. As to some of the details: The general definitions of the trigonometric functions of any angle are given at once by means of a rectangular coördinate system, and the functions of acute angles are considered as special cases. There are two chapters

on the solution of triangles, the first without logarithms. In dealing with the inverse functions the author starts with $y=\sin x$ and considers the relation $x=\text{arc}\sin y$. In the reviewer's opinion first year students will more quickly grasp the functional relations involved if (1) $y=\text{arc}\sin x$ is the starting point, and $x=\sin y$ is used only as an aid in plotting (1). No mention is made of the alternative notation $\sin^{-1} x$. It would have been well to call attention to this, since it is of such frequent occurrence in the mathematical literature. Tables are not included, except one page of natural functions to three places for instructional purposes. The ground covered is no more than what is essential for practical applications, and to serve as a background for further work in mathematics.

Coming to Part II, we find that the author adheres to his plan of unification, as the following chapter headings indicate: Cartesian coördinates, Equations of loci, Straight lines, Functions and their graphs, Solution of nth degree equations, Rate of change, Differentiation of polynomials, Parabolas, Circles, Differentiation of algebraic functions, Central conics, Curve tracing, Integration, Parametric equations, Transformation of coördinates, Polar coördinates, Complex numbers. These topics are dovetailed into a unified whole, and at the same time the treatment of each is adequate and comprehensive; *e.g.*, as sub-headings for the chapter on nth degree equations we find: Rational integral equations, solution of general quadratic equation, number of roots of an equation, relation between roots and coefficients, integral roots, Descartes's rule of signs, rational roots, irrational roots; and for the chapter on differentiation of a polynomial: Derivatives, differentiation formulas, geometric interpretation of a derivative, equations of tangents to curves, sign of the derivative, velocity, maxima and minima, applications of the theory of maxima and minima, Newton's method of approximating irrational roots. The remaining chapters are equally well rounded out.

A noteworthy feature of both Part I and Part II is the arrangement of exercises. These are divided into groups A, B, C, and D. Group A consists of easy problems calling for an obvious application of the theory; group B is similar to group A, but more difficult; group C calls for some ingenuity; group D, not present in every set of exercises, contains many problems whose solution constitutes an extension of the preceding theory.

The reviewer is anxious to avoid leaving with his readers the impression that *First Year College Mathematics* is just two more texts. It can come pretty close to the ideal solution of the problem of unified mathematics; for the student who takes only one course it provides substantial information in a wide range of useful topics; for the student who intends to go on to further work, it provides a rock-bottom foundation. Furthermore, the selection and arrangement of material; the concise and scholarly presentation; the attractive drawings; the sparing but effective use of display and bold-faced type; all these combine to form a set-up in textbook design and construction which is distinctly superior.

R. L. Jeffery

Construction, Classification, and Census of Magic Squares of Order Five. By A. L. Candy, Author and Publisher. 1003 H Street, Lincoln, Nebraska, 1938. 5+141 pages. $1.00.

The purpose of this book is to analyze and classify completely the entire set of magic squares of order five. The term "magic square" is here used in the classical sense to mean a square array of the positive integers from 1 to 25 in which the sum of the numbers in each row, in each column, and in each of the two principal diagonals is the same, namely 65.

The author divides all magic squares of order five into seven general classes and then discusses certain sub-sets of these classes, called "types." He shows how magic squares of each of these types can be built up and succeeds in enumerating completely the squares belonging to some of the types. The labor involved in finding the number of squares in the remaining types proved to be prohibitive, so that their number was found by making a statistical estimate. Thus Professor Candy finds that there are 12,860,440 essentially different magic squares of order five. This number, although for the most part an estimate, is far from being a mere guess but seems to be based on fairly good statistical evidence.

Although much has been written on magic squares a great deal of the literature is difficult of access, so that an author may easily overlook some previous work. This has happened in some parts of the present book. For instance, the author's "Mixed squares Class II, Type I" are precisely the bordered squares discussed in detail by B. Violle in 1838. Both Violle and the author consider 26 cases but the number of squares corresponding to each case agree in 7 cases only. In 14 instances the author's number is greater than Violle's and in 5 cases smaller. However, in the example on page 80 the first pair of rows add to 63 and 67 instead of to 65 each. With this correction the number of squares for this case is reduced to 28 which agrees with Violle's number.

In the chapter on pandiagonal squares every example given is of the well known uniform step type. There are, however, other pandiagonal squares derivable from the uniform step squares by certain transformations. The total number of pandiagonal squares of order 5 is known to be 3600. Of these 3600 squares no one can be obtained from any other by rotations or reflections.

The author has succeeded in bringing to light a number of very interesting and unusual squares which may well be worth further investigation. Furthermore, the book offers a challenge to those interested in the subject to make a new attempt to determine the exact number of magic squares of order five. As the book amply illustrates, the solution of this problem is not easy, at least by the methods used there.

A number of misprints were noted, but they are of a rather obvious nature, so that it seems unnecessary to list them.

G. E. Raynor

The Mathematics of Investment. By P. R. Rider. New York, Farrar and Rinehart, Inc., 1938. 162 pages. $2.00.

This book is an excellent text for a one-semester first course in the mathematics of investment where the prerequisite work is at least one year of high school algebra. The text presents the important and fundamental conceptions in a clear, concise, and accurate fashion.

The first chapter contains a number of useful algebraic formulas and explanations of certain necessary processes and manipulations which are pertinent in the study of the mathematics of investment.

The second chapter is devoted to the consideration of interest and discount. It is pleasing to note a clearly stated and correct definition of interest and discount. Students should not have the usual confusion in understanding what discount is. The author, if not the only author, is among the very few who defines clearly and correctly an equation of value in terms of the "Principle of Equivalence."

The third chapter is devoted to the consideration of annuities. We do not find here the usual number of many unnecessary special case formulas. In fact, he very quickly arrives at the treatment of an annuity where the payments are made p times per period and the interest is converted m times per period. Such a concise and rather inclusive treatment is, in the reviewer's opinion, superior as it enables the student to consider problems as special cases of a single algebraic result. Capitalized cost is also presented in this chapter. In the reviewer's opinion, it would have been better to discuss this conception in chapter four where other applications of annuities are discussed.

The fourth chapter contains an adequate treatment of the important applications of annuities. Here we find a clear presentation of amortization, sinking funds, and the valuation of a mine. We also find here a discussion of certain aspects of the concept, depreciation. The author discusses a few of the common methods used for obtaining the periodic depreciation charge and building up the so-called depreciation fund. However, as is the case in many other texts on the mathematics of investments, depreciation itself is not defined. In the reviewer's opinion, it should have been emphasized that depreciation is in fact a statistical problem.

The fifth chapter contains a brief but adequate treatment of building and loan associations.

The sixth chapter contains a brief but adequate treatment of bonds and their valuation. The reviewer is of the opinion that stocks and their valuation should also have been discussed in this chapter.

The seventh chapter is devoted to an excellent brief presentation of the elementary principles of probability, the meaning of expectation, and the nature of a mortality table. Enough is said about the application and use of probability so that the student will have a clear understanding of the meaning of the various so-called rates of mortality as given in a mortality table.

The eighth chapter is devoted to a brief discussion of the meaning of an

endowment, a life annuity, deferred life annuity, temporary life annuity, and a forborne temporary life annuity. The author also defines the necessary commutation columns which are useful to simplify computation. Of course, much of the theory of life annuities has been omitted and justifiably so. However, the reviewer feels that the connection between a life annuity and an old age pension should have been specifically mentioned.

The ninth chapter is devoted to a brief discussion of the meaning of life insurance and a few of the common types of life insurance policies. It is shown how the annual and single premiums may be found by the use of commutation columns. Loading and one form of valuation is briefly discussed. The reviewer feels that one very important subject, the valuation of life insurance policies, is inadequately treated.

A very good set of tables is bound with the book. The clarity of the exposition is most pleasing. The text appears to be a very teachable one and in the reviewer's opinion is a valuable and worth while addition to the set of available texts.

F. M. Weida

Manual of Mathematics and Mechanics. By G. R. Clements and L. T. Wilson. First Edition. New York, McGraw-Hill Book Company, 1937. 5+263 pages. $2.50.

This compendium of mathematical information contains 85 pages of numerical tables, 145 pages of formulas, figures and curves pertaining to the usual topics in mathematics covered in the high school and the first two years of college, a little about vectors, and 44 pages of the principles and formulas of mechanics. There are 22 numerical tables including the usual common and Napierian logarithms; direct and inverse trigonometric, exponential, and hyperbolic functions, mostly to four figures after the decimal; squares, cubes, etc. of numbers; compound interest, annuity, and mortality tables; values of elliptic and probability integrals; conversion tables; and binomial coefficients.

The table of integrals contains 362 formulas for general or definite integrals. There are 15 short articles on series, 4 on the formulas of plane and solid geometry, 37 on algebra including such subjects as the cubic equation, probability, determinants, interest, annuities, and complex numbers. Plane and spherical trigonometry are given 17 articles and plane and solid analytic geometry 39 articles. There are 9 pages of figures of curves and surfaces followed by 44 articles on differential and integral calculus, consisting mostly of formulas including ordinary differential equations. Vectors are discussed in 9 articles and the formulas and principles of mechanics in 32. There are 15 pages of formulas giving the location of the centroid, moment and product of inertia, and radius of gyration of certain plane and solid figures. The book closes with 12 articles on the properties of materials followed by 4 pages of figures and formulas for shear, moment, and deflection of loaded beams. An index of 4 pages adds to the usefulness of the volume.

Certain minor omissions seem obvious to the reviewer, such as: the polar equations of the cissoid, the witch, and particularly the strophoid, which are so much simpler than the rectangular or parametric equations, the parabolic cylinder $y^2=ax+bz$, and especially the formula for area $\frac{1}{2}\int(xdy-ydx)$, since so many parametric equations are included.

It seems that the third form for $a_x=v_x(dv_x/dx)$ should be given in Art. 176 because it is so frequently useful, and that the special cases in which the formula $M=I\alpha$ can be used in addition to the one given in Art. 191 should be stated. The reviewer is inclined to question the use of the term constant density for uniform density and to ask for a definition of an element of a hoop.

These are all unimportant points, however, in view of the extent and detail of the book. It is certainly an excellent handbook for the student of college mathematics and mechanics. It will be useful, also, as a syllabus for college students majoring in mathematics.

J. B. REYNOLDS

Segmental Functions, Text and Tables. By C. K. Smoley. First Edition. Scranton, Pa., C. K. Smoley and Sons, 1937. 43+255 and 91+34 pages.

In this book are bound together in two parts, with non-consecutive paging, a number of tables some of which are new, while others are abstracted from previous publications by the author. The new tables, covering 255 pages, have to do with what are called segmental functions because of their relation to a segment of a circle.

If A, M, C, and R stand respectively for the lengths of arc, middle ordinate, chord, and radius of the circle of a circular segment, seven of these segmental functions are defined by

$$a = A/R, \qquad c = C/R, \qquad m = M/R, \qquad \alpha = A/C,$$
$$\beta = M/C, \qquad \gamma = M/A, \qquad \sigma = MR/A^2.$$

The logarithms of these functions are given for each minute to five figures after the decimal for values of the central angle, subtended by the segment, from 0° to 180°. There is also a table, for the same range, of the function $f=F/R^2$ in which F is the area of the segment. Separate tables give the logarithm of a for each 10″ over the first 10° of the central angle and the natural values of a over the wider range.

The first 43 pages are devoted to a discussion of segmental functions and explanation of their uses. The mathematician is surprised to read that the central angle, ϕ, is not included in the definition of segmental functions and then find that the very first one is the ratio of the arc to the radius. In fact all the eight functions listed are nothing more than trigonometric functions defined by $a=\phi$, $c=2\sin\frac{1}{2}\phi$, $m=\text{vers}\,\frac{1}{2}\phi$, $\alpha=\frac{1}{2}\phi/\sin\frac{1}{2}\phi$, $\beta=\frac{1}{2}\tan\frac{1}{4}\phi$, $\gamma=\text{vers}\,\frac{1}{2}\phi/\phi$, $\sigma=\text{vers}\,\frac{1}{2}\phi/\phi^2$, $f=\frac{1}{2}(\phi-\sin\phi)$. In view of this, it seems unlikely that these tables will be of service except to a few persons engaged in certain narrowly technical fields.

J. B. REYNOLDS

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

Beginning with the next issue of the MONTHLY, the department of Mathematics Clubs will be under the editorship of Professor E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, New Jersey. All future correspondence and new material should be sent to him. Professor Hildebrandt brings to us vital interest and a fresh and stimulating viewpoint. The retiring editors bespeak for him the same sincere cooperation of the one hundred fifty mathematics clubs in the United States and Canada which has made the last three years such worth while ones for the editors.

It has been impossible to print all programs received, but all have been filed as a permanent record of undergraduate activities. These files, together with the Stunt Library, the modest beginnings of a File of Undergraduate Publications and Student Papers will be placed in the hands of the new editor.

Let every club make a sincere effort to send reports for 1937–38 to the editor before October first. Promptness in this matter is essential to the successful planning of the work for the new year. Particularly be sure that a correct list of officers for 1938–39 is sent direct to this department as soon as they are elected. The greater the cooperation of the many clubs, the better can this department serve its purpose.

BOOKS

34. *Excursions in Mathematics*, by Ernst R. Breslich. Chicago, The Orthovis Company, 1938. 48 pages. Illustrated. $1.20. A clever aid to visualization of the problems of area and volume.
35. *Poetry of mathematics and other essays*, by David Eugene Smith. New York, *Scripta Mathematica*, 1927. 96 pages. $1.00. Presents the subjects considered from a cultural viewpoint; quite worthy of consideration.
36. *Portraits of eminent mathematicians with their biographies*, by David Eugene Smith. New York, *Scripta Mathematica*. Portfolio I (12 mathematicians) $3.00. Portfolio II (13 mathematicians) $3.00. A real addition to any club library and interesting from the point of instruction in history of mathematics and in building visual familiarity with well known men in various fields.
37. *Mathematical Nuts*, by S. I. Jones, Nashville, Tenn. 1936. $3.50.
38. *Mathematical Wrinkles*, by S. I. Jones, Nashville, Tenn. 1930. $3.00. This and the preceding book by the same author are filled with interesting and unusual bits for the recreational side of club work.
39. *Mathematical Excursions and Essays*, by W. W. R. Ball. New York, Macmillan, 10th edition, 1922. $3.50. A favorite for many years as its frequent republication testifies. It is one of the books which will soon show wear if left on an open shelf.

CLUB REPORTS
1936–37

Kappa Mu Epsilon, Kansas State Teachers College, Pittsburg

President, Sylvia Smith; Vice-President, J. Shaw; Secretary, Aileene Kingsbury; Treasurer, Colleen Hawkins; Corresponding Secretary, Professor W. H. Hill. The chapter held open meeting each month with programs which always included musical numbers in addition to the talks on mathematical topics. Topics included: Mathematics of the American Aborigines; Number theory among the ancients; Philosophy of mathematics; Japanese mathematics; Algebraic and transcendental numbers; The development of geometry. In addition to these, several old favorites appeared which are always of interest to the new students. Among the social meetings the most noteworthy was the celebration of the fifth anniversary of the founding of the chapter. The original charter list bore twenty-seven names. Since that time one hundred and seventy-three names have been added. The celebration was attended by over seventy, many alumni returning for the event.

Mathematics Club, University of California at Los Angeles

This group which welcomes all students interested in mathematics holds open meetings each month. The topics are chosen with regard to their appeal to a large group. Paper folding, Trisection of an angle, and similar topics are favorites. Experiences in Canadian and English universities illustrated the different types of instruction in mathematics. Picnics took members from the sea side to the high mountains.

Kappa Mu Epsilon, Kansas State Teachers College, Emporia

The chapter held ten program meetings in addition to the social affairs which serve to increase interest among the members. Topics discussed included: Telescopes and their construction; Einstein, the man; Dependence of physics on mathematics; Educated man; Mathematics and art; Mathematics and chemistry. One meeting was devoted to reports of the Annual Meetings of the Kansas Association of Mathematics Teachers and the Kansas Section of the Mathematical Association of America. Special discussions concerned prophecies with reference to the future of mathematics in the secondary schools.

Delta x, University of Toledo

President, N. Farley; Vice-Presidents, A. Hiltner, E. Keefer; Secretary, Georgia M. Miller; Treasurer, E. Tom. The eighth year of this active group started off with a discussion of applications of conic sections. Each meeting is announced by a mimeographed sheet generally distributed. The first for this year closed with the modest statement: "Come and see why Delta x is the biggest club of its kind in the United States." The new feature of the year's program was a review of current mathematics magazines which pertain to undergraduates. Book reviews, problems, recreations, all received attention. The membership of over seventy made a high average of attendance. Membership is open to all students taking a course in calculus or who have completed such a course.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems. [By error problems for solution were omitted in this number of the MONTHLY. E. J. M.]

SOLUTIONS

E 303 [1937, 598]. *Proposed by J. E. Trevor, Cornell University.*

Given fixed rectangular axes, x and y, and a straight line T perpendicular to the x-axis but capable of displacement parallel to itself. One end of a straight line segment A of constant length a is pivoted to T at the point where T cuts the x-axis. The other end of A makes sliding contact with the y-axis. One end of a straight line segment B of constant length b is pivoted at the origin. The other end maintains sliding contact with T. The segments A and B, produced if necessary, intersect at P.

Find the rational algebraic equation of the locus of P. Describe the graph of this equation according as a is less than, equal to, or greater than b. Find the point at which the locus cuts the x-axis in the cases where a equals or exceeds b.

Solution by the proposer.

(a) Let T be the line $x=t$, and put

$$\alpha = \pm\sqrt{a^2 - t^2}, \qquad \beta = \pm\sqrt{b^2 - t^2}.$$

Then, by similar triangles, if x and y are the coördinates of P,

$$\frac{\alpha}{t} = \frac{y}{t - x}, \qquad \frac{\beta}{t} = \frac{y}{x};$$

whence, successively eliminating y and x,

$$x = \frac{\alpha t}{\alpha + \beta}, \quad y = \frac{\alpha\beta}{\alpha + \beta}, \tag{1}$$

which are parametric equations of the locus, with parameter t. Elimination of t yields the equation of the locus,

$$(a^2-b^2)^2(x^2+y^2)^4-2b^2(x^2+y^2)\left[(a^2-b^2)x^2+a^2y^2\right]\left[(a^2-b^2)x^2+(a^2+b^2)y^2\right] + b^4\left[(a^2-b^2)x^2+a^2y^2\right]^2=0. \tag{2}$$

This equation contains only terms of degrees eight, six, and four.

(b) From its equation it appears that the curve is symmetrical with regard

to the coördinate axes. Except for the case $a=b$, it lies in a finite part of the plane, with no asymptotes or parabolic branches. The origin is a four-fold point, isolated for $b<a$, but with two tangents, each counted twice, when $a<b$. The points $(\pm b, 0)$ are double points, isolated for $a<b$, but with two tangents for $b<a$. Aside from the four-fold point at the origin, the curve cuts the y axis in four real points,

$$\left(0, \frac{\pm ab}{a \pm b}\right).$$

Figures 1 and 2 sketch the curve for the cases $a=2$, $b=1$, and $a=1$, $b=2$. In

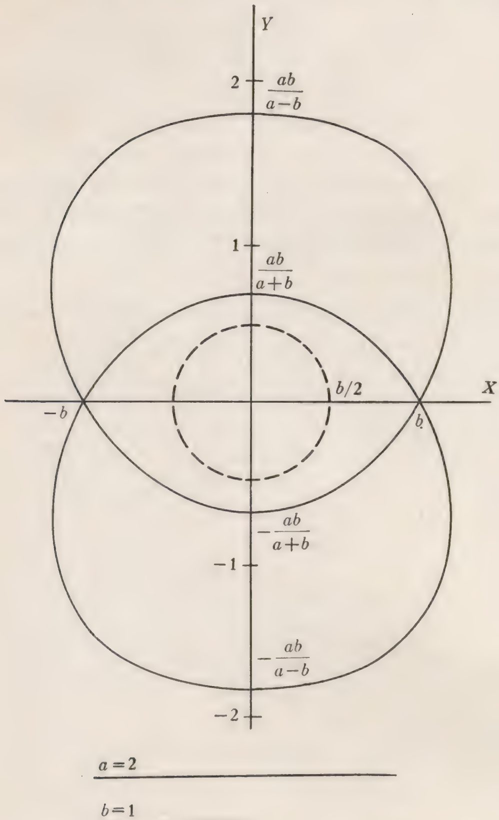

FIG. 1

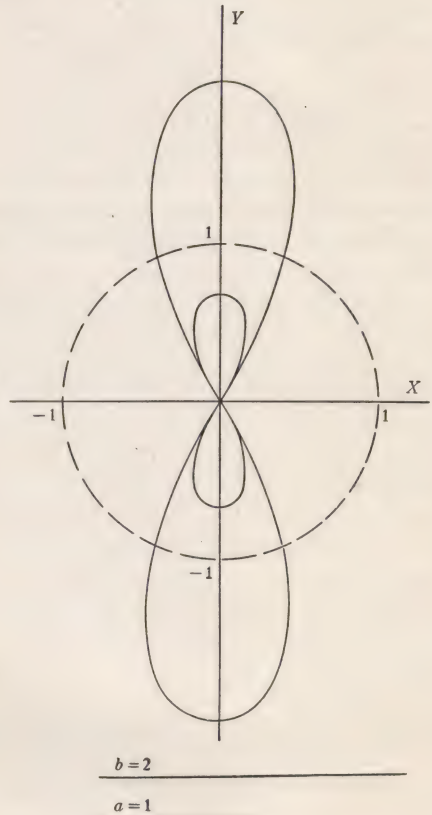

FIG. 2

each figure the dashed circle is the locus for $a=b$. In this case $\alpha=\beta$, and the equations (1) become

$$x = \frac{t}{2}, \qquad y = \frac{\pm\sqrt{a^2 - t^2}}{2},$$

whence, eliminating t,

$$x^2 + y^2 = \left(\frac{a}{2}\right)^2.$$

The locus is a circle of radius $a/2$. For this case (2) reduces to

$$- a^6(4x^2 + 4y^2 - a^2)y^4 = 0.$$

(c) When $a \geqq b$ and a approaches b, the curve cuts the x axis at $x = \pm b$. But when $a = b$ the point of intersection jumps to the intersection $x = \pm b/2$ of the axis with the circular graph for that case.

Also solved by Fred Discepoli.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch side.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3877. *Proposed by Louis Weisner, Hunter College, New York.*

Show that it is possible to construct a set S consisting of an infinite number of positive, and an infinite number of negative integers, which has the following properties: (1) If the integers of S are arranged in ascending order of magnitude, the difference between consecutive positive integers, and the difference between consecutive negative integers of S, tend to infinity. (2) If $f(x)$ is *any* non-constant polynomial with integral coefficients, S contains an infinite number of the integers represented by $f(x)$ for integral values of x. The problem becomes trivial if the first requirement is omitted: we could then take S as the set of all integers.

3878. *Proposed by V. Thébault, Le Mans, France.*

A convex quadrilateral is circumscribed about a circle. Show that there exists a straight line segment with ends on opposite sides dividing both the perimeter and the area into equal parts. Show that the straight line passes through the center of the inscribed circle. Consider the converse.

3879. *Proposed by V. Thébault, Le Mans, France.*

Show that it is possible to determine a plane section, limited by three faces, of a tetrahedron which divides both the surface and volume into equal parts. Show that the plane passes through the center of the inscribed sphere.

3880. *Proposed by W. R. Bennett, Bell Telephone Laboratories, New York.*

Show that the following identity involving continuants is true for any positive integer n. The notation is that used in Chrystal's *Algebra*, vol. II, p. 466:

$$K\{1, 2\lambda \sin \pi/2n, 2\lambda \sin 3\pi/2n, \cdots, 2\lambda \sin (2n-1)\pi/2n, 1\}$$
$$= 2[\lambda - ie^{\pi i/2n}][\lambda - ie^{3\pi i/2n}] \cdots [\lambda - ie^{(2n-1)\pi i/2n}].$$

3881. *Proposed by W. R. Bennett, Bell Telephone Laboratories, New York.*

Show that the following identity is true for any positive integer n, where the notation is that given in Chrystal's *Algebra*, vol. II, p. 466:

$$K\left\{\begin{matrix} 1 \quad , & \cos^2 \pi/2n, & \cdots, & \cos^2 (n-1)\pi/2n \\ 1, \ \lambda \sin \pi/2n, & \lambda \sin 3\pi/2n, & \cdots, & \lambda \sin (2n-1)\pi/2n \end{matrix}\right\}$$
$$= 2^{n-1}[\lambda - ie^{\pi i/2n}][\lambda - ie^{3\pi i/2n}] \cdots [\lambda - ie^{(2n-1)\pi i/2n}].$$

SOLUTIONS

3791 [1936, 378]. *Proposed by I. M. Sheffer, Pennsylvania State College.*

Let P_1, P_2, P_3 be three points on a circle C, so situated that P_2 and P_3 are on opposite sides of the diameter through P_1, and chords P_1P_2 and P_1P_3 make angles less than 45° with this diameter. Form a lattice-work in the plane with P_1P_2 and P_1P_3 as adjacent sides of a lattice parallelogram. Then of all the lattice points in the plane, only P_1, P_2, P_3 are in or on C.

Solution by E. P. Starke, Rutgers University.

The hypotheses on P_1, P_2, P_3 are equivalent to the statement that the inscribed triangle $P_1P_2P_3$ is acute-angled. Let L, M, N be the points P_1, P_2, P_3 in any order. Let Q be the vertex opposite L of the parallelogram $LMNQ$, and let Q' be the corresponding vertex of an inscriptible convex quadrilateral $LMNQ'$. Since angle Q' is the supplement of the acute angle Q (=angle L), angle Q' is greater than angle Q and point Q is outside the circle C.

All lattice points on the extended sides of triangle LMN are obviously outside C. All lattice points within the angle MLN (and hence within or on the sides of the vertical angle of angle Q) are outside C, for the line connecting any one of them with Q (outside the circle) when produced will intersect the chord MN (at a point in or on C). All lattice points within the vertical angle of angle MLN are outside C, for the line connecting any one of them with L (on C) when produced will intersect the chord MN (at a point in C).

Since L was any one of the points P_1, P_2, P_3, the proposition is established for all lattice points of the plane.

Solved also by the proposer.

Editorial Note. The proposer gave a proof by analytic geometry. It is simpler to use vector methods. Let P_1 be the origin of vectors, and let $\mathbf{b}_2$, $\mathbf{b}_3$, $\mathbf{c}$ be the vectors of P_2, P_3, C, where the last point is the center of circle (C) on which lie the three given distinct points. Any lattice point (n_2, n_3) is given by the vector $n_2\mathbf{b}_2+n_3\mathbf{b}_3$, where n_2 and n_3 are any two integers. A necessary and sufficient condition that all lattice points but the three given lie outside (C) is that the three points $(1, 1)$, $(-1, 1)$, $(1, -1)$ lie outside (C). Denote by r the distance of a lattice point from C. Then

$$
\begin{aligned}
r^2 - \mathbf{c}^2 &= (n_2\mathbf{b}_2 + n_3\mathbf{b}_3 - \mathbf{c})^2 - \mathbf{c}^2 \\
&= n_2^2\mathbf{b}_2^2 + 2n_2n_3\mathbf{b}_2\cdot\mathbf{b}_3 + n_3^2\mathbf{b}_3^2 - 2\mathbf{c}\cdot(n_2\mathbf{b}_2 + n_3\mathbf{b}_3) \\
&= (n_2^2 - n_2)\mathbf{b}_2^2 + (n_3^2 - n_3)\mathbf{b}_3^2 + 2n_2n_3\mathbf{b}_2\cdot\mathbf{b}_3.
\end{aligned} \tag{1}
$$

The last result follows from

$$2c = \mathbf{b}_2^2\mathbf{b}_2' + \mathbf{b}_3^2\mathbf{b}_3', \qquad \mathbf{b}_i\cdot\mathbf{b}_i' = 1, \qquad \mathbf{b}_i\cdot\mathbf{b}_j' = 0, \qquad i \neq j,$$

see 3752 [1937, 405]. A necessary and sufficient condition that $(1, 1)$, $(-1, 1)$, $(1, -1)$ lie outside is given by (1) in the three inequalities

$$\mathbf{b}_2\cdot\mathbf{b}_3 > 0, \qquad \mathbf{b}_2^2 > \mathbf{b}_2\cdot\mathbf{b}_3, \qquad \mathbf{b}_3^2 > \mathbf{b}_2\cdot\mathbf{b}_3. \tag{2}$$

We now show that if (2) is true, every point but the given points $(0, 0)$, $(1, 0)$, $(0, 1)$ lies outside (C). For this purpose we may exclude the six points already mentioned; for the remaining points not both of the first two terms of the right member of the last equation in (1) can be zero. Hence by (2) we have from (1) for these points

$$r^2 - \mathbf{c}^2 > [(n_2 + n_3)^2 - (n_2 + n_3)]\mathbf{b}_2\cdot\mathbf{b}_3 \geqq 0, \tag{3}$$

where the final inequality results from the fact that n_2+n_3 is an integer and thus the bracket expression is never negative. This completes the proof. The inequalities (2) say that each angle of triangle $P_1P_2P_3$ is less than a right angle.

3792 [1936, 435]. *Proposed by F. Morley, Johns Hopkins University.*

A square is divided into n^2 unit squares, like a chess-board. Any two horizontal lines and any two vertical lines form a rectangle. We count a square as a rectangle. Each rectangle has a breadth b, less than or equal to its length. There is one rectangle of breadth n, namely the original square. Prove that there are 2^3 rectangles of breadth $n-1$, 3^3 of breadth $n-2, \cdots, n^3$ of breadth 1.

Deduce the formula

$$1^3 + 2^3 + \cdots + n^3 = [n(n+1)/2]^2.$$

Solution by Roy MacKay, Eastern New Mexico Junior College.

There are only $k+1$ ways in which $n-k$ unit squares can be taken along one edge to form a rectangle of length $n-k$. Therefore the number of rectangles of width $n-k$ where the width is measured in an east-west (=EW) direction is $(k+1)[(k+1)+k+(k-1)+ \cdots +2+1]=(k+1)(k+1)(k+2)/2$. Of this number $(k+1)^2$ are squares. Hence the total number of rectangles of width $n-k$ is $2(k+1)(k+1)(k+2)/2-(k+1)^2=(k+1)^3$.

To derive the desired formula we have but to consider the unit squares as formed by $(n+1)$ lines in the EW direction meeting $(n+1)$ lines in the NS direction. There are ${}_{n+1}C_2$ pairs of lines in each direction and hence $[{}_{n+1}C_2]^2 = [n(n+1)/2]^2$ rectangles in all.

Solved also by W. B. Campbell, G. N. Garrison, E. P. Starke, and C. W. Trigg.

3793 [1936, 435]. *Proposed by V. Thébault, Le Mans, France.*

Three parallels drawn from the midpoints of the sides BC, CA, AB of a triangle cut again the nine-point circle in α, β, γ. (a) Show that the circles with centers α, β, γ passing respectively through the feet A', B', C' of the altitudes AA', BB', CC' of the given triangle cut each other in pairs in three points of a straight line Δ and also pass through a point Q. (b) When the direction of the parallels varies, prove that the line Δ passes through a fixed point and that the point Q describes the nine-point circle of ABC. (c) Generalize the theorem by replacing the nine-point circle by the pedal circle of a point D in the plane of the triangle ABC and the points A', B', C' by the orthogonal projections of D on BC, CA, AB.

Solution by the proposer.

The solution of this problem will be deduced from the following theorem:

In the plane of the given triangle ABC the point D is arbitrarily selected. Denote by (ω) the circle with center ω through A', B', C', the orthogonal projections of D on the sides of ABC. Let A_2, B_2, C_2 be the projections of the vertices of ABC on the straight line Δ passing through D with an arbitrarily given direction; let α, β, γ be the center of circles (α), (β), (γ) circumscribing the triangles $A'B_2C_2$, $B'C_2A_2$, $C'A_2B_2$. Then (α), (β), (γ), (ω) intersect in a point Q on the circumcircle of the triangle $\alpha\beta\gamma$.

For the proof of this theorem we introduce three auxiliary circles (α'), (β'), (γ') with the diameters DA, DB, DC, and consequently with centers α', β', γ' at the midpoints of these segments. These circles circumscribe respectively the polygons $AA_2B'C'D$, $BB_2C'A'D$, $CC_2A'B'D$. With D as center of inversion and any convenient radius the inverses of the circles (α'), (β'), (γ') are three straight lines, while Δ inverts into itself. These four straight lines form a complete quadrilateral determining four triangles whose vertices are respectively the inverses of A', B', C'; B_2, A_2, C'; B', A_2, C_2; A', B_2, C_2. The circumcircles of these four triangles of the complete quadrilateral meet in the Miquel point M. These circumcircles are the inverses of (ω), (γ), (β), (α); hence these latter four circles must meet in the inverse Q of M.

We now make use of the following theorem:

The circumcenter of the triangle whose vertices are three of the intersections of three given circles, and the circumcenter of the triangle whose vertices are the remaining three intersections, are isogonal conjugate points with respect to the triangle whose vertices are the centers of the three given circles. Neuberg, *Nouvelle correspondence mathématique*, 1880–8.

Here we have the three circles (α), (β), (γ) which meet in the point Q while the other three intersections A_2, B_2, C_2 lie on Δ. Then one triangle of intersection degenerates to the point Q while the remaining triangle degenerates to the straight line Δ. Hence with respect to the triangle $\alpha\beta\gamma$ the point Q and the point at infinity on the perpendicular to Δ through Q are isogonal conjugate points. From this follows that the circumcircle of triangle $\alpha\beta\gamma$ is a fifth circle passing through Q and this completes the proof.

Editorial Note. The proof seems simpler without the use of the general theorem of Neuberg. Let U, V, W be the centers of circles (U), (V), (W) passing through the point M and intersecting in pairs in T, R, S on a straight line l, where the common chord of (U) and (V) is MT, etc. The side UV of triangle UVW is perpendicular to the common chord MT at its midpoint. Since the midpoints of chords MR, MS, MT lie on a straight line parallel to l, the feet of the perpendiculars from M to the sides of UVW must lie on this parallel to l. It now follows that the circumcircle of UVW passes through M. This simple theorem suffices for this part of the proof, but we shall note a slight extension. Let I be any point on l and let I_u, I_v, I_w, be the inverses of I with respect to (U), (V), (W). Then the circle through I_u, I_v, I_w, passes also through M, as is easily seen by inversion with respect to any suitable circle with I as center, noting that the inverse of I_u is the center of the inverse of (U). When I is at infinity on l we have merely the first theorem.

We now return to the complete quadrilateral with the Miquel point M. There are four circles which intersect in M any three of which also intersect in pairs on a side in the three vertices on that side. Hence by the above theorem the four centers of these circles are such that the circle through any three of these centers passes also through M. Thus we have eight circles through M. Now consider a point D on the side Δ of the quadrilateral and the inverses of D with respect to the three circles intersecting on Δ. The circle through these three inverse points also passes through M, and we now have nine circles through M. If for each position of D on Δ we invert this system with respect to D for the case in which we obtain the original figure of the problem, we have at once the theorem at the head of the above solution and also the following result:

Let the inverses of D with respect to (α), (β), (γ), (ω) be α_d, β_d, γ_d, ω_d. Then there are four circles each of which passes through three of these last four points and also through Q. We have now nine circles through Q. There are three other circles in the figure of the problem. The center γ lies on a circle (γ_1) through α', β', C_1, C'. For $\alpha'\gamma$, $C_1\gamma$ are perpendicular respectively to $C'A_2$, DA_2; and, as the direction of Δ varies with D fixed, A_2 describes the circle (α'). Hence γ describes a circle through α' and C_1. Similarly, we see that this circle passes through β' and C_1. When Δ has the position DC', the circle (γ) is the point circle C' and γ is at C', and this completes the proof. If D is at the orthocenter of ABC, the circles (α_1), (β_1), (γ_1), (ω) coincide in the nine-point circle of ABC; thus the circumcircle of $\alpha\beta\gamma$ and (ω) coincide.

3794 [1936, 435]. *Proposed by R. S. Underwood, Texas Tech. College, Lubbock, Tex.*

Prove that the series

$$\sum_{n=1}^{\infty} \left| \frac{\sin n\theta}{n} \right|$$

is divergent unless θ is an integral multiple of π.

Solution by H. Tate, McGill University, Montreal.

Abel's test states that, if $u_1+u_2+u_3+\cdots$ is a series such that the absolute value of the sum of the first n terms is less than a fixed number for all values of n, and, if $v_1, v_2, v_3, \cdots$ is a monotonically decreasing sequence of positive numbers which approach zero as n becomes infinite; then

$$u_1v_1 + u_2v_2 + u_3v_3 + \cdots$$

is a convergent series.

If $\theta \neq 2k\pi$, where k is an integer, then

$$\sum_{i=1}^{n} \cos i\theta = \frac{\sin n\theta/2}{\sin \theta/2} \cos (n+1)\theta/2, \tag{1}$$

and the absolute value of this sum is less than $1/|\sin \theta/2|$. Hence the series

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} \tag{2}$$

is convergent for all values of $\theta \neq 2k\pi$. Again

$$\left| \frac{\sin n\theta}{n} \right| \geqq \frac{\sin^2 n\theta}{n} = \frac{1}{2n}[1 - \cos 2n\theta],$$

and it has been proved that $\sum \cos 2n\theta/n$ is convergent if $\theta \neq k\pi$. Since $\sum n^{-1}$ is divergent, the series of the problem is divergent.

Solved also by J. H. Curtiss, M. Goldberg, E. P. Starke, and the proposer.

Editorial Note. Curtiss stated that the solution of the problem is contained in a theorem by P. Fatou, *Bulletin de la Société Mathématique de France*, vol. 41, 1913, pp. 47–53, which says that, if the series $\sum a_n \sin n\theta$ is absolutely convergent for a value $\theta_0 \neq 0$, mod π, then $\sum |a_n|$ is convergent provided that $|a_n|$ never increases. He also stated that a simple proof of Fatou's theorem is given in Zygmund's new book on trigonometric series p. 134, and pp. 3, 4.

A proof of Abel's test is given in Goursat-Hedrick, vol. 1, pp. 348, 349. In Tonelli's *Serie Trigonometriche* a proof of the Fatou theorem for $\sum a_n \cos n\theta$ is given on page 26, and the theorem of the problem is stated as an illustration of the theorems on pages 28 and 29.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

Dr. C. L. Alsberg of Stanford University has been appointed director of the Giannini Foundation of Agricultural Economics at the University of California.

Dr. M. T. Bird served as representative of the Mathematical Association at the Semicentennial Celebration of the Utah State Agricultural College June 5–7, 1938.

Garrett Birkhoff of Harvard University has been promoted to an assistant professorship in mathematics.

Dr. Julia W. Bower of Connecticut College has been promoted to an assistant professorship.

Dr. Melvin Dresher has been appointed to an instructorship at Michigan State College.

At Harvard University the following men have been appointed as instructors and tutors in the Division of Mathematics, on part time, for 1938–1939: R. F. Clippinger, A. D. Hestenes, D. T. McClay, Abraham Spitzbart. Dr. J. W. Green of the University of California and Dr. D. T. Perkins of Yale University have been appointed Benjamin Peirce instructors in mathematics for the academic year 1938–39.

Dr. E. J. Finan of the Catholic University of America has been promoted to the rank of assistant professor, effective October 1938.

The Rockefeller Foundation has extended the fellowship now held by Dr. A. S. Householder through the academic year 1938–39. He will continue his work in mathematical biophysics with Professor N. Rashevsky at the University of Chicago.

Assistant Professor R. R. McDaniel of the Virginia State College, Ettrick, Va., has been promoted to an associate professorship.

C. W. Trigg, formerly at Cumnock College, has been appointed an instructor at Los Angeles Junior College.

Dr. E. P. Wigner, who resigned a year ago from the faculty of Princeton University to become professor of theoretical physics at the University of Wisconsin, has been recalled to Princeton as Thomas D. Jones professor of mathematical physics.

The following seventy-one doctorates with mathematics as major subject were conferred during 1937 in the United States and Canada. The university, month in which the degree was conferred, minor subject (other than mathematics), and title of the dissertation are given in each case if available.

O. B. Ader, Kentucky, August. *Concerning affine invariants of convex regions.*

C. B. Allendoerfer, Princeton, January. *The embedding of Riemann spaces in the large.*

G. F. Alrich, Maryland, June. *Hyperconformal transformations.*

M. L. Bishop, New York, June, minor in physics. *On the algebra of finite square matrices with real quaternion elements.*

Archie Blake, Chicago, August. *Canonical expressions in Boolean algebra.*

R. P. Boas, Jr., Harvard, June. *Iterated Stiltjes transform.*

J. W. Calkin, Harvard, June. *Partial differential operators.*

Harold Chatland, Chicago, August. *The aysmptotic Waring problem for homogeneous polynomial summands.*

H. R. Cooley, New York, February, minor in physics. *Some systems of quasi-linear partial differential equations with identical principal parts.*

C. H. Denbow, Chicago, June. *A generalized form of the problem of Bolza.*

C. H. Dieckmann, California (Berkeley), May. *A study of the surfaces generated by a point that moves in certain relationships to two fixed lines of space.*

F. W. Dresch, California (Berkeley), August. *Applications of index numbers to the study of general economic equilibria.*

Melvin Dresher, Yale, June. *Multi-groups. A generalization of the notion of groups.*

W. A. Dwyer, Nebraska, June, minor in physics. *On certain fundamental identities due to Uspensky.*

F. G. Fisher, California (Berkeley), May. *The representation of quadric surfaces upon a space of nine dimensions.*

A. S. Galbraith, Harvard, June. *Second-order differential equations with irregular singular points of special types.*

F. C. Gentry, Illinois, June, minor in physics. *Groups of Cremona transformations in space of ternary type.*

M. J. Gottlieb, Washington University (St. Louis), June, minor in physics. *An investigation of polynomials orthogonal on a finite or enumerable set of points.*

Anne M. C. Grant, Bryn Mawr, June. *Asymptotic transitivity on surfaces of variable negative curvature.*

Louis Green, Chicago, August. *Systems of quadrics associated with a point of a surface.*

O. H. Hamilton, Texas, June, minor in economics. *Non-unique solutions of first order ordinary differential equations.*

B. A. Hausmann, Yale, June. *Quasi-groups.*

A. E. Heins, Massachusetts Institute of Technology, February. *The multidimensional operational calculus.*

A. S. Householder, Chicago, June. *The dependence of a focal point upon curvature in the calculus of variations.*

D. H. Hyers, California Institute of Technology, June, minor in physics. *Integrals and functional equations in linear topological spaces.*

J. E. Ikenberry, Cornell, June. *An involutorial transformation with a multiple correspondence on lines joining conjugate points.*

S. B. Jackson, Harvard, June. *Differential properties in the large of spherical curves.*

E. C. Kennedy, Rice, June. *Fuchsian groups of genus two.*

R. H. Kershner, Johns Hopkins, June. *The addition of convex curves and the Riemann zeta-function.*

L. A. Knowler, Iowa, May, minor in applied mathematics and physics. *Actuarial aspects of recent old age security legislation.*

D. B. Kohlmetz, Ohio State, December. *Certain problems of a special character in convex functions.*

O. E. Lancaster, Harvard, June. *Non-linear algebraic difference equations.*

Caroline A. Lester, Wisconsin, June. *A determination of the automorphisms of certain algebraic fields.*

L. L. Lowenstein, Cornell, September. *Linear equations with an infinity of unknowns.*

H. A. Luther, Iowa, July, minor in physics. *A third order irregular boundary value problem.*

Dorothy Manning, Stanford, June. *On simply transitive groups with transitive Abelian subgroups of the same degree.*

A. N. Milgram, Pennsylvania, June. *Decomposition and dimensions of closed sets in* R''.

Clemmer Mitchell, Cincinnati, June. *On Euler summability and one of its generalizations.*

Virginia Modesitt, Illinois, June, minor in astronomy. *Some singular properties of conformal transformations between Riemannian spaces.*

A. P. Morse, Brown, June. *Convergence in variation and related topics.*

N. A. Moscovitch, Chicago, March. *Studies of the inverse problem of the calculus of variations.*

Z. I. Mosesson, Harvard, June. *Maximal sequences of polynomials.*

D. C. Murdoch, Toronto, June. *Fundamental exponents in the theory of algebraic numbers.*

C. J. Nesbit, Toronto, June. *On the regular representations of algebras.*

A. C. Olshen, Iowa, May, minors in applied mathematics and commerce. *Transformations of the Pearson type III distribution.*

K. L. Palmquist, Kentucky, August. *Ideals in a quaternion algebra and Hermitian forms.*

E. W. Paxson, California Institute of Technology, June, minor in physics. *Analysis in linear topological spaces.*

Sallie E. Pence, Illinois, June, minor in astronomy. *The configuration of the double points of cubics of a pencil.*

Paul Pepper, Cincinnati, June. *An application of geometry of numbers to a generalization of a continued fraction.*

B. J. Pettis, Virginia, June. *On integration in vector spaces.*

M. H. L. Pryce, Princeton, June. major in mathematical physics. *On the theory of light quanta.*

W. T. Puckett, Jr., Virginia, June. *Some properties of cyclic elements of higher order.*

Harriet Rees, Chicago, August. *Ideals in cubic and certain quartic fields.*

Erwin Ritter, New York, June, minor in physics. *The problem of Plateau for a Schwarz chain.*

J. H. Schotland, New York, June. *An analysis of methods of plane curve fitting.*

Hyman Serbin, Pittsburgh, February. *Non-commutative polynomials.*

Ruth G. Simond, Michigan, August. *Relations between certain continuous transformations.*

M. F. Smiley, Chicago, August. *Discontinuous solutions for the problem of Bolza in parametric form.*

E. C. Stopher, Iowa, February, minor in applied mathematics. *Interrrelations of a family of operators on point sets and their canonical representation.*

W. C. Taylor, Wisconsin. *Asymptotic formulas for the Whittaker function.*

R. M. Thrall, Illinois, June, minor in physics. *Metabelian groups and trilinear forms.*

Annita Tuller, Bryn Mawr, June. *The measure of transitive geodesics on certain three-dimensional manifolds.*

F. A. Valentine, Chicago, March. *The problem of Lagrange with differential inequalities as added side conditions.*

R. H. Valentine, Massachusetts Institute of Technology, June, minor in physics. *Travel-time curves in oblique structures.*

R. W. Wagner, Michigan, June. *Multiple-valued functions in matrix space.*

Henry Wallman, Princeton, June. *Lattices and topological spaces.*

J. V. Wehausen, Michigan, August. *Studies in linear topological spaces.*

F. P. Welch, Illinois, June, minor in physics. *Singular non-linear first order differential equations containing a parameter.*

E. T. Welmers, Michigan, February. *Set functions and measurability conditions.*

L. S. Winton, Duke, June. *Compatible integro-differential systems.*

Shaun Wylie, Princeton, June. *Duality and intersection in general complexes.*

THE JOURNAL OF DOCUMENTARY REPRODUCTION

The following notice concerning a new quarterly may be important to mathematicians:

"The American Library Association, Chicago, announces the publication, beginning this month, of a quarterly *Journal of Documentary Reproduction.*"

The significance of this brief announcement is not apparent and, were it not amplified, would fail to reveal a forecast of the library of the future. The new *Journal* presents a glimpse of a library which will have at its readers' disposal the resources of the great libraries of the world—the rare and inaccessible books, manuscripts, and newspapers heretofore available alone to those few who could journey to distant museum or library.

Today the preservation of civilization's records are more certain because of the achievements of microphotography and related techniques in reproducing written and pictorial documents at low cost. The earliest printed books, unique manuscripts and folios, the crumbling files of news-

papers great and small, and other priceless historical records, are now being recorded. . . . Research manuscripts are reproduced for use of scholars for a few cents a page.

The *Journal of Documentary Reproduction* will keep librarians, archivists, scientists, and others abreast of ever-changing developments in the reproduction of documents. . . .

Outlining the scope of the new *Journal* in its first issue, Winter 1938, a thick 128-page periodical, Charles E. Rush, associate librarian of Yale University Library and chairman of the editorial board, points out that "attention will be given to the several allied processes of reproduction and duplication adapted to the dissemination of information and not at present within the range of the printing press." The *Journal* will supplement and keep up to date *Microphotography for Libraries*, the first two volumes on the subject ever published, which were edited by M. Llewellyn Raney, director of the University of Chicago Libraries, and published by the Association in 1936 and 1937.

Other features of the first issue of the *Journal* include a résumé of technical research by the Bureau of Standards; announcement of the newly organized American Documentation Institute, a central agency for microfilming service; a technical section with descriptions of new equipment, new services, and applications of new techniques; news of projects under way in libraries, museums, and elsewhere; and an extensive bibliography of photographic methods of documentary reproduction.

The *Journal*, published by the American Library Association, is edited by its Committee on Photographic Reproduction of Library Materials with Vernon D. Tate, chief of the Division of Photographic Reproduction of the National Archives at Washington as managing editor.The editorial board includes Mr. Rush as chairman; Mr. Fussler; Robert C. Binkley, professor of history, Columbia University; Keyes D. Metcalf, director, Harvard University Libraries; George A. Schwegmann, Jr., director, Union Catalogue, Library of Congress; and Paul Vanderbilt, librarian, Pennsylvania Museum of Art. The subscription price is $3 a year.

EXAMINATION QUESTIONS FOR THE FIRST WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION, APRIL 16, 1938

MORNING SESSION: 9:00 A.M. to 12:00 NOON.

1. A solid is bounded by two bases in the horizontal planes $z=h/2$ and $z=-h/2$, and by such a surface that the area of every section in a horizontal plane is given by a formula of the sort Area $= a_0z^3 + a_1z^2 + a_2z + a_3$ (where as special cases some of the coefficients may be 0). Show that the volume is given by the formula $V = 1/6\, h[B_1 + B_2 + 4M]$, where B_1 and B_2 are the areas of the bases, and M is the area of the middle horizontal section. Show that the formulas for the volume of a cone and of a sphere can be included in this formula when $a_0=0$.

2. A can buoy is to be made of three pieces, namely, a cylinder and two equal cones, the altitude of each cone being equal to the altitude of the cylinder. For a given area of surface, what shape will have the greatest volume?

3. If a particle moves in the plane, we may express its coordinates x and y as functions of the time t. If $x=t^3-t$ and $y=t^4+t$, show that the curve has a point of inflection at $t=0$, and that the velocity of the moving particle has a maximum at $t=0$.

4. A lumberman wishes to cut down a tree whose trunk is cylindrical and whose material is uniform. He will cut a notch, the two sides of which will be planes intersecting at a dihedral angle θ along a horizontal line through the axis of the cylinder. If θ is given, show that the least volume of material is cut out when the plane bisecting the dihedral angle is horizontal.

5. Evaluate the following limits:

(*a*) $$\lim_{n\to\infty}\frac{n^2}{e^n}.$$ (*b*) $$\lim_{x\to 0}\frac{1}{x}\int_0^x (1+\sin 2t)^{1/t}\,dt.$$

6. A swimmer stands at one corner of a square swimming pool and wishes to reach the diago-

nally opposite corner. If w is his walking speed and s is his swimming speed ($s<w$), find his path for shortest time. [Consider two cases: (*a*) $w/s<\sqrt{2}$, and (*b*) $w/s>\sqrt{2}$.]

7. TAKE EITHER (*a*) OR (*b*).

(*a*) Show that the gravitational attraction exerted by a thin homogeneous spherical shell at an external point is the same as if the material of the shell were concentrated at its center.

(*b*) Determine all the straight lines which lie upon the surface $z=xy$, and draw a figure to illustrate your result.

AFTERNOON SESSION: 2:00 P.M. to 5:00 P.M.

8. TAKE EITHER (*a*) OR (*b*).

(*a*) Let A_{ik} be the cofactor of a_{ik} in the determinant

$$d = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}.$$

Let D be the corresponding determinant with a_{ik} replaced by A_{ik}. Prove $D=d^3$.

(*b*) Let $P(y)=Ay^2+By+C$ be a quadratic polynomial in y. If the roots of the quadratic equation $P(y)-y=0$ are a and b ($a\neq b$), show that a and b are roots of the biquadratic equation $P[P(y)]-y=0$. Hence write down a quadratic equation which will give the other two roots, c and d, of the biquadratic. Apply this result to solving the following biquadratic equation:

$$(y^2 - 3y + 2)^2 - 3(y^2 - 3y + 2) + 2 - y = 0.$$

9. Find all the solutions of the equation

$$y\,y'' - 2(y')^2 = 0$$

which pass through the point $x=1$, $y=1$.

10. A horizontal disc of diameter 3 inches is rotating at 4 revolutions per minute. A light is shining at a distant point in the plane of the disc. An insect is placed at the edge of the disc furthest from the light, facing the light. It at once starts crawling, and crawls so as always to face the light, at 1 inch per second. Set up the differential equation of motion, and find at what point the insect again reaches the edge of the disc.

11. Given the parabola $y^2=2mx$. What is the length of the shortest chord that is normal to the curve at one end?

12. From the center of a rectangular hyperbola a perpendicular is dropped upon a variable tangent. Find the locus of the foot of the perpendicular. Obtain the equation of the locus in polar coordinates, and sketch the curve.

13. Find the shortest distance between the plane $Ax+By+Cz+1=0$ and the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$. (For brevity, let

$$h = 1/\sqrt{A^2 + B^2 + C^2} \text{ and } m = \sqrt{a^2A^2 + b^2B^2 + c^2C^2}.)$$

State algebraically the condition that the plane shall lie outside the ellipsoid.

Note. Chairmen of mathematics departments may obtain copies of the examination questions, as long as the supply lasts, by writing for them to Professor W. D. Cairns, Oberlin, Ohio.

Back Numbers Wanted

For a limited number of copies of back issues of the MONTHLY, the Association will pay (or will credit on members' dues) for any of the following at forty cents a copy:

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1914, Any single numbers except December
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1930, Jan.
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